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Original Research Article

m-Component Reliability Model in Bayesian Inference on Modified Weibull Distribution

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Abstract

In order to produce more flexible models in the reliability theory field, the Bayesian inference of m-component reliability model with the non-identical-component strengths for modified Weibull distribution under the progressive censoring scheme is considered. One of the key benefits is the generality of this model, so it includes some cases studied previously, such as multi-component stress-strength model with one and two non-identical-component and stress-strength models. In addition, the study of progressive censored data discussed in this paper is critical in many practical situations. The problem is considered in three cases: when the two common parameters for strengths and stress variables are unknown, known, and general. In each case, the approximation methods, such as the MCMC and Lindley's approximation, are used to consider the m-component stress-strength parameter. The Monte Carlo simulation study compares the performance of different methods—finally, a demonstration of how the proposed model may be utilized to analyse real data sets.

Keywords: Multi-component stress-strength reliability; Lindley's approximation; MCMC method; Progressive censoring scheme;

Nomenclature & Units

AL	Average lengths
CP	Coverage percentages
$F(x)$	Cumulative distribution function (CDF)
GPA	Grade point average
$h(x)$	Hazard rate function (HRF)
HPD	Highest posterior density
L	Likelihood function
$MCMC$	Markov Chain Monte Carlo
MLE	Maximum likelihood estimation
$R_{s,k}$	m-component reliability parameter
$MCSS$	m-component stress-strength
MSE	Mean squared error
MWD	Modified Weibull distribution
$R_{s,k}$	Multi-component reliability parameter
n	Number of observed samples
k	Number of strength variable components
s	Number of strength variable components exceeding stress variable
N	Total number of sample
β	Modified Weibull parameter
γ	Modified Weibull parameter
λ	Modified Weibull parameter
$f(x)$	Probability density function (PDF)
P-P $plot$	Probability-probability plot
$\{X_{1:n:N}, \cdots, X_{n:n:N}\}$	Progressive censored sample
$\{R_1, \cdots, R_n\}$	Progressive censoring scheme
$R = P(X > Y)$	Reliability parameter
X	Strength variable
Y	Stress variable
WPP	Weibull probability plot

1. Introduction

This paper studies the Bayesian inference of an MCSS parameter with non-identical-component strengths for the MWD under a progressive censoring scheme. This model is so general because some cases can be derived from there. The m-component reliability parameter can be converted to a multi-component with one and two non-identical-component cases or the stress-strength parameter in special cases. Besides, the progressive censoring scheme can be converted to a type-II censoring scheme and complete data case. Also, the MWD can be converted to Weibull distribution, type-I extreme value distribution, Rayleigh distribution, and exponential distribution. Accordingly, about 24 cases are studied automatically. Several basic research studies have been

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carried out in this field. Regarding the stress-strength model, the progressive censoring in exponential distribution is discussed in [1]. The model was also analyzed in several other studies, including [2] (Weibull distribution), [3] (power Lindley distribution), and [4] (MWD). Regarding the multi-component reliability model with one strength variable, the Kumaraswamy distribution in progressive censoring data is considered in [5]. The model was also analyzed in several other studies, including [6] (unit Gompertz distribution), [7] (a general class of inverted exponentiated distribution), [8] (Topp-Leone distribution), [9] (Log-normal distribution), [10] (unit Burr III distribution), and [11] (power Lindley distribution). Each of the papers above has unique properties and compensates for the objections and problems in previous research. The m-component reliability model and progressive censoring data are considered in [12] in the modified Weibull extension distribution. The modified Weibull extension distribution can be converted to Chen and Weibull distributions, and the MWD can be converted to Weibull, type-I extreme value, Rayleigh, and exponential distributions. Thus, we can manoeuvre on the purpose and motivation of the present work.

The two most common censoring schemes are Type-I and Type-II censoring schemes, which can be mixed to form a hybrid censoring scheme. The progressive censoring scheme was introduced since none of the above schemes allowed the removal of active units during the experiment. The scheme is discussed in detail using an excellent monograph [13]. A nonparametric estimation of the family of risk measures based on the progressive censoring scheme has recently been considered in [14]. Also, the statistical inference in the Burr type XII lifetime model based on progressive randomly censored data is studied in [15]. The progressive censoring scheme can be described as follows: Consider an experiment in which N units are placed on a life test. During the test, R_1 units are randomly removed from the test at the time of the first failure, R_2 units are randomly removed from the test at the time of the second failure, and so on. R_n units are randomly removed from the test at the time of the n-th failure. In this scheme, the progressive sample is $\{X_{1:n:N}, \cdots, X_{n:n:N}\}$, and the progressive censoring scheme is $\{R_1, \cdots, R_n\}$, such that $R_1 + \cdots + R_n + n = N$. In what follows, the progressive censored sample is expressed as $\{X_1, \cdots, X_n\}$. The joint PDF of failure times $X_1 < \cdots < X_n$ with a continuous PDF $f(\cdot)$ and CDF $F(\cdot)$ is given by:

$$f(x_1, \cdots, x_n) \propto \prod_{i=1}^{n} f(x_i)(1 - F(x_i))^{R_i}, \tag{1}$$

Fig. 1 illustrates a schematic representation of the progressive censoring scheme.

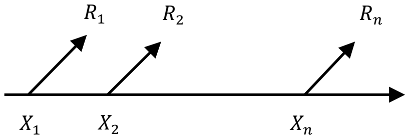


Figure 1. Schematic representation of progressive scheme

The statistical inference of the reliability parameter $R = P(X > Y)$ has attracted the attention of researchers in reliability theory. Here, variables X and Y denote strength and stress, respectively. The m-component reliability model with $\boldsymbol{k} = (k_1, k_2, \dots, k_m)$ components have recently been developed in [16] as:

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{i=1}^{m} \binom{k_i}{p_i}\right) \int_{-\infty}^{\infty} \prod_{i=1}^{m} ((1 - F_i(y))^{p_i} (F_i(y))^{k_i - p_i}) dF_Y(y), \tag{2}$$

where k_i components are of $i, i = 1, \dots, m$ type, and $F_i(\cdot)$ is the CDF of the strengths of the i-th type components. Under this condition, it is assumed that all components are exposed to a common stress Y with $F_Y(\cdot)$ CDF. Therefore, the system is reliable only if at least $\boldsymbol{s} = (s_1, \dots, s_m)$ of $\boldsymbol{k}$ strength components exceed the stress. This model has recently been considered for the modified Weibull extension distribution in progressive censored data in [12], which has also been discussed in this paper for the MWD. The model is regarded as general because some cases can be derived from there:

- $\boldsymbol{k} = (k_1, k_2, 0, \cdots, 0) \Rightarrow R_{\boldsymbol{s},\boldsymbol{k}}$ with two non-identical-component cases
- $\boldsymbol{k} = (k, 0, \cdots, 0) \Rightarrow R_{s,k}$ case
- $\boldsymbol{k} = (1, 0, \dots, 0) \Rightarrow R = P(X < Y)$ case

A new modification of the Weibull distribution is proposed in [17] by multiplying the Weibull distribution. Some of its properties are studied using MLE and WPP in [17]. The PDF, CDF, and HRF of the MWD are as follows:

$$f(x) = \beta(\gamma + \lambda x)x^{\gamma-1}e^{\lambda x}e^{-\beta x^{\gamma} e^{\lambda x}}, x > 0, \tag{3}$$

$$F(x) = 1 - e^{-\beta x^{\gamma} e^{\lambda x}}, x > 0, \tag{4}$$

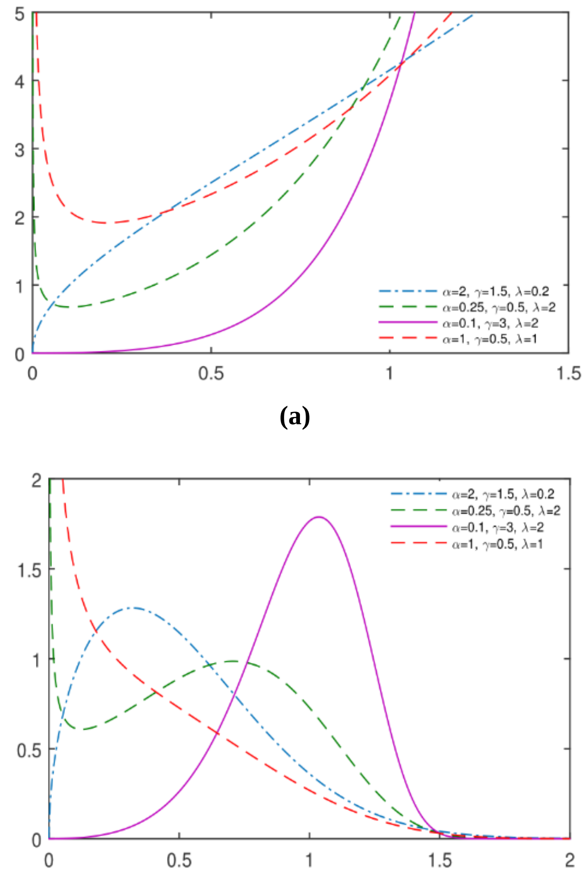
$$h(x) = \beta(\gamma + \lambda x)x^{\gamma-1}e^{\lambda x}, x > 0, \tag{5}$$

where $\beta > 0$, $\gamma, \lambda \geq 0$, and at most one of γ, λ is equal to zero.

- The Weibull distribution is a special case for $\lambda = 0$.
- The type-I extreme value distribution is a special case for $\gamma = 0$.
- The Rayleigh distribution is a special case for $\lambda = 0$ and $\gamma = 2$.
- The exponential distribution is a particular case for $\lambda = 0$ and $\gamma = 1$.

The PDF of the MWD can be decreasing, unimodal, or decreasing, then unimodal-shaped. The HRF, on the other hand, can be increasing or bathtub-shaped. The

MWD can analyze many real-world datasets thanks to its flexibility. For example, the reliability of a reverse osmosis system in water treatment using the MWD has recently been evaluated in [18]. Fig. 2 displays some possible shapes of the PDF and the HRF for the MWD. This paper obtained the Bayesian inference of $R_{s,k}$ based on the progressive censored sample, where X and Y are two independent random variables from the MWD.



(a)

(b)

Figure 2. The shape of hazard rate **(a)** and probability density **(b)** functions of MWD. (blue: $\alpha = 2, \gamma = 1.5, \lambda = 0.2$, green: $\alpha = 0.25, \gamma = 0.5, \lambda = 2$, violet: $\alpha = 0.1, \gamma = 3, \lambda = 2$, red: $\alpha = 1, \gamma = 0.5, \lambda = 1$.)

The remainder of this paper is structured as follows: In Section 2, the Bayesian inference of $R_{s,k}$ is obtained for the unknown common γ and λ parameters using the MCMC method, and the HPD credible intervals are constructed for $R_{s,k}$. In Section 3, the Bayesian inference of $R_{s,k}$ is obtained for the known common γ and λ parameters using the MCMC and Lindley's approximation methods, and the HPD credible intervals are constructed for $R_{s,k}$. In Section 4, the Bayesian inference of $R_{s,k}$ is obtained for the general case. Section 5 provides simulation and data analysis. Finally, Section 6 presents the conclusions.

2. Inference on $R_{s,k}$ with Unknown Common γ and λ Parameters

If $X_1 \sim MWD(\beta_1, \gamma, \lambda)$, …, $X_m \sim MWD(\beta_m, \gamma, \lambda)$, and $Y \sim MWD(\beta, \gamma, \lambda)$ are independent random variables, then the MCSS parameter $R_{s,k}$ can be obtained from Eqs. (3) and (4) as:

$$R_{s,k} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \int_0^\infty \beta\,(\gamma + \lambda y) \times y^{\gamma-1} e^{\lambda y} e^{-y^\gamma e^{\lambda y}\left(\sum_{l=1}^m \beta_l p_l + \beta\right)} \times \prod_{l=1}^m \left(1 - e^{-\beta_l y^\gamma e^{\lambda y}}\right)^{k_l - p_l} dy$$

By putting: $t = y^\gamma e^{\lambda y}$, we have

$$= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \int_0^\infty \beta\, t^{\sum_{l=1}^m \beta_l p_l + \beta} \times \prod_{l=1}^m \left(1 - t^{\beta_l}\right)^{k_l - p_l} dt$$
$$= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{k_1 - p_1}{q_1} \cdots \binom{k_m - p_m}{q_m} (-1)^{\sum_{l=1}^m q_l} \beta \times \int_0^1 t^{\sum_{l=1}^m \beta_l (p_l + q_l) + \beta - 1} dt \tag{6}$$
$$= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \binom{k_1 - p_1}{q_1} \cdots \binom{k_m - p_m}{q_m} \frac{(-1)^{\sum_{l=1}^m q_l} \beta}{\sum_{l=1}^m \beta_l (p_l + q_l) + \beta}.$$

The likelihood function can be constructed based on the following samples of the stress and strength variables:

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, X_l = \begin{pmatrix} X_{11}^{(l)} & \cdots & X_{1k_l}^{(l)} \\ \vdots & \ddots & \vdots \\ X_{n1}^{(l)} & \cdots & X_{nk_l}^{(l)} \end{pmatrix}, l = 1, \ldots, m,$$

where $\{Y_1, \ldots, Y_n\}$ is a progressively censored sample from $MWD(\beta, \gamma, \lambda)$ with the $\{n, S_1, \ldots, S_n\}$ censoring scheme. Besides, $\{X_{i1}^{(l)}, \ldots, X_{ik_1}^{(l)}\}$, $i = 1, \ldots, n$, $l = 1, \ldots, m$ are l progressive censored samples from $MWD(\beta_l, \gamma, \lambda)$ with schemes $\{k_l, R_1^{(l)}, \ldots, R_{k_l}^{(l)}\}$. The likelihood function of the parameters can be obtained as follows:

$$L(\beta_1, \ldots, \lambda_m, \beta, \gamma, \lambda | \text{data}) \propto \prod_{i=1}^n \left(\prod_{l=1}^m \left(\prod_{j_l=1}^{k_l} f_l\left(x_{ij_l}^{(l)}\right) \times \left(1 - F_l\left(x_{ij_l}^{(l)}\right)\right)^{R_j^{(l)}} \right) f(y_i) (1 - F(y_i))^{S_i}. \right. \tag{7}$$

This section studies the Bayesian inference of $R_{s,k}$ under squared error loss functions where $\beta_l, \ldots, \beta_m, \beta, \gamma$, and λ are independent random variables. Based on the observed censoring samples, the joint posterior density function is expressed as follows:

$$\pi(\beta_1,\dots,\beta_m,\beta,\gamma,\lambda|\text{data}) \propto L(\text{data}|\beta_1,\dots,\lambda_m,\beta,\gamma,\lambda) \times\times \left(\prod_{l=1}^{m}\pi(\beta_l)\right)\pi(\beta)\pi(\gamma)\pi(\lambda), \tag{8}$$

where

$$\pi(\beta_l) \propto \beta_l^{a_l-1}e^{-b_l\beta_l},\ \ a_l,b_l>0,\ l=1,\dots,m, \tag{9}$$

$$\pi(\beta) \propto \beta^{a_{m+1}-1}e^{-b_{m+1}\beta}, a_{m+1},b_{m+1}>0, \tag{10}$$

$$\pi(\gamma) \propto \gamma^{c_{m+1}-1}e^{-d_{m+1}\gamma},\ c_{m+1},d_{m+1}>0, \tag{11}$$

$$\pi(\lambda) \propto \lambda^{e_{m+1}-1}e^{-f_{m+1}\lambda},\ \ e_{m+1},f_{m+1}>0. \tag{12}$$

From Eq. (8), since the Bayes estimate cannot be obtained in the closed form, it should be approximated using the MCMC method. Thus, the posterior PDFs of $\beta_1,\dots,\beta_m,\ \beta,\gamma,$ and λ can be derived from Eq. (8) as follows:

$$\beta_l|\gamma,\lambda,\text{data} \sim \Gamma\left(nk_l+a_l, b_l+\sum_{i=1}^{n}\sum_{j=1}^{k_l}\left(R_j^{(l)}+1\right)\times\left(x_{ij}^{(l)}\right)^{\gamma}e^{\lambda x_{ij}^{(l)}}\right),\ l=1,\dots,m, \tag{13}$$

$$\beta|\gamma,\lambda,\text{data} \sim \Gamma\left(n+a_{m+1}, b_{m+1}+\sum_{i=1}^{n}(S_i+1)\,y_i^{\gamma}e^{\lambda y_i}\right), \tag{14}$$

$$\pi(\gamma|\beta_1,\dots,\beta_m,\beta,\lambda,\text{data}) \propto \prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j=1}^{k_l}\left(\left(x_{ij}^{(l)}\right)^{\gamma}\left(\gamma+\lambda x_{ij}^{(l)}\right)\right)\times\prod_{i=1}^{n}\left(y_i^{\gamma}(\gamma+\lambda y_i)\right)\times\gamma^{c_{m+1}-1}\times e^{-\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j=1}^{k_l}\beta_l\left(R_j^{(l)}+1\right)\left(x_{ij}^{(l)}\right)^{\gamma}e^{\lambda x_{ij}^{(l)}}}\times e^{-\beta\sum_{i=1}^{n}(S_i+1)y_i^{\gamma}e^{\lambda y_i}}\times e^{-d_{m+1}\gamma}, \tag{15}$$

$$\pi(\lambda|\beta_1,\dots,\beta_m,\beta,\gamma,\text{data}) \propto \prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j=1}^{k_l}\left(\gamma+\lambda x_{ij}^{(l)}\right)\times\prod_{i=1}^{n}(\gamma+\lambda y_i)\times e^{-\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j=1}^{k_l}\beta_l\left(R_j^{(l)}+1\right)\left(x_{ij}^{(l)}\right)^{\gamma}e^{\lambda x_{ij}^{(l)}}}\times e^{-\beta\sum_{i=1}^{n}(S_i+1)y_i^{\gamma}e^{\lambda y_i}}\times\lambda^{e_{m+1}-1}e^{-\lambda\left(f_{m+1}-\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j=1}^{k_l}x_{ij}^{(l)}-\sum_{i=1}^{n}y_i\right)}. \tag{16}$$

As shown, samples should be generated from the posterior PDFs of γ and λ using the Metropolis-Hastings method as they are unknown PDFs. For this purpose, the following Gibbs sampling algorithm is proposed:

1. Start by selecting an initial value $(\beta_{1(0)},\dots,\beta_{m(0)}\ ,\beta_{(0)},\gamma_{(0)},\lambda_{(0)})$.
2. Set $t=1$.
3. Generate value $\gamma_{(t)}$ from $\pi(\gamma|\beta_{1(t-1)},\dots,\beta_{m(t-1)},\ \ \beta_{(t-1)},\lambda_{(t-1)},\text{data})$ using the Metropolis-Hastings method, with $N\left(\gamma_{(t-1)},1\right)$ as the proposal distribution.
4. Generate value $\lambda_{(t)}$ from $\pi(\lambda|\beta_{1(t-1)},\dots,\beta_{m(t-1)},\ \ \beta_{(t-1)},\gamma_{(t-1)},\text{data})$ using the Metropolis-Hastings method, with $N\left(\lambda_{(t-1)},1\right)$ as the proposal distribution.
5. m+4. Generate value $\beta_{l(t)}$ from $\Gamma\left(nk_l+a_l, b_l+\sum_{i=1}^{n}\sum_{j=1}^{k_l}\left(R_j^{(l)}+1\right)\left(x_{ij}^{(l)}\right)^{\gamma(t-1)}e^{\lambda(t-1)x_{ij}^{(l)}}\right)$.

m+5. Generate value $\beta_{(t)}$ from $\Gamma\left(n+a_{m+1},b_{m+1}+\sum_{i=1}^{n}(S_i+1)\,y_i^{\gamma(t-1)}e^{\lambda(t-1)y_i}\right)$.

m+6. Evaluate

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1}\dots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\dots\sum_{q_m=0}^{k_m-p_m}\binom{k_1}{p_1}\dots\binom{k_m}{p_m}\times\binom{k_1-p_1}{q_1}\dots\binom{k_m-p_m}{q_m}\times\frac{(-1)^{\sum_{l=1}^{m}q_l}\beta_{(t)}}{\sum_{l=1}^{m}\beta_{l(t)}(p_l+q_l)+\beta_{(t)}}. \tag{17}$$

m+7. Set $t=t+1$.

m+8. Repeat T times, steps 3 - m+7.

Therefore, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$, under the squared error loss functions is:

$$\hat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MB} \ \ = \frac{1}{T}\sum_{t=1}^{T}R_{(t)\boldsymbol{s},\boldsymbol{k}}\,. \tag{18}$$

Also, the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be constructed using the method proposed in [19] as follows. Order $R_{(1)\boldsymbol{s},\boldsymbol{k}},\dots,R_{(T)\boldsymbol{s},\boldsymbol{k}}$ as $R_{((1)\boldsymbol{s},\boldsymbol{k})}<\dots<R_{((T)\boldsymbol{s},\boldsymbol{k})}$, and construct all $100(1-\eta)\%$ confidence intervals of $R_{\boldsymbol{s},\boldsymbol{k}}$ as $\left(R_{((1)\boldsymbol{s},\boldsymbol{k})},R_{(([T(1-\eta)])\boldsymbol{s},\boldsymbol{k})}\right)$, …, $\left(R_{(([T\eta])\boldsymbol{s},\boldsymbol{k})},R_{(([T])\boldsymbol{s},\boldsymbol{k})}\right)$, where $[T]$ symbolizes the largest integer less than or equal to T. The HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ is the shortest-length confidence interval.

3. Inference on $R_{s,k}$ with Known Common γ and λ Parameters

This section obtains Bayesian estimation and the corresponding credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ under the squared error loss function. Assuming that $\beta_1,\dots,\beta_m$ and β follow the independent gamma distributions as prior distributions, similar to Section 2, the posterior PDFs of the parameters are obtained as follows:

$$\beta_l|\gamma,\lambda,\text{data} \sim \Gamma\left(nk_l+a_l, b_l+\sum_{i=1}^{n}\sum_{j=1}^{k_l}\left(R_j^{(l)}+1\right)\times\left(x_{ij}^{(l)}\right)^{\gamma}e^{\lambda x_{ij}^{(l)}}\right),\ l=1,\dots,m, \tag{19}$$

$$\beta|\gamma,\lambda,\text{data} \sim \Gamma\left(n+a_{m+1}, b_{m+1}+\sum_{i=1}^{n}(S_i+1)\,y_i^{\gamma}e^{\lambda y_i}\right), \tag{20}$$

Therefore, the Gibbs sampling algorithm can be implemented as follows:

1. Start by selecting an initial value $\left(\beta_{1(0)},\dots,\beta_{m(0)},\beta_{(0)}\right)$
2. Set $t=1$.
3. 3 - m+2. Generate value $\beta_{l(t)}$ from $\Gamma\left(nk_l+a_l, b_l+\sum_{i=1}^{n}\sum_{j=1}^{k_l}\left(R_j^{(l)}+1\right)\left(x_{ij}^{(l)}\right)^{\gamma(t-1)}e^{\lambda(t-1)x_{ij}^{(l)}}\right)$.

   m+3. Generate value $\beta_{(t)}$ from $\Gamma\left(n+a_{m+1},b_{m+1}+\sum_{i=1}^{n}(S_i+1)\,y_i^{\gamma(t-1)}e^{\lambda(t-1)y_i}\right)$.

m+4. Evaluate

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{k_1-p_1}{q_1} \cdots \binom{k_m-p_m}{q_m} \times \frac{(-1)^{\sum_{l=1}^{m} q_l} \beta_{(t)}}{\sum_{l=1}^{m} \beta_{l(t)}(p_l+q_l)+\beta_{(t)}}. \quad (21)$$

m+5. Set $t = t + 1$.

m+6. Repeat T times, steps 3 - m+5.

Therefore, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ under the squared error loss functions is:

$$\hat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MB} = \frac{1}{T}\sum_{t=1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}. \quad (22)$$

Also, the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be constructed using the method introduced in [19].

Lindley's approximation [20] is one of the most important numerical methods to obtain the approximate Bayes estimation of a parameter. Bayesian estimation of $U(\Theta)$ can be derived under the squared error loss functions as:

$$\mathbb{E}(u(\theta)|\text{data}) = \frac{\int u(\theta) e^{Q(\theta)} d\theta}{\int e^{Q(\theta)} d\theta}, \quad (23)$$

where $Q(\theta) = \rho(\theta) + \ell(\theta)$, $\rho(\theta)$ and $\ell(\theta)$ are the logarithms of the prior density θ and log-likelihood functions, respectively. Eq. (23) is approximated in [20] as follows:

$$\mathbb{E}(u(\theta)|\text{data}) = u + \frac{1}{2}\sum_i \sum_j \left(u_{i,j} + 2u_i\rho_j\right)\sigma_{i,j} + \frac{1}{2}\sum_i \sum_j \sum_k \sum_p \ell_{i,j,k}\, \sigma_{i,j}\sigma_{k,p} u_p \big|_{\theta=\hat{\theta}}, \quad (24)$$

where $\theta = (\theta_1, \ldots, \theta_m)$, $i,j,k,p = 1, \ldots, m$, $\hat{\theta}$ is the MLE of θ, $u = u(\theta)$, $u_i = \partial u/\partial\theta_i$, $u_{i,j} = \partial^2 u/\left(\partial\theta_i\, \partial\theta_j\right)$, $\ell_{i,j,k} = \partial^3\ell/\left(\partial\theta_i\, \partial\theta_j\, \partial\theta_k\right)$, $\rho_j = \partial\rho/\partial\theta_j$, and $\sigma_{i,j} = (i,j)^{\text{th}}$ element in the inverse of the matrix $\left[-\ell_{i,j}\right]$, all evaluated at the MLE of parameters. By rewriting Eq. (24) for $m+1$ parameters, we have:

$$\hat{u}^{Lin} = u + \left(\sum_{i=1}^{m+1} u_i\, d_i + d_{m+2} + d_{m+3}\right) + \frac{1}{2}\sum_{i=1}^{m+1} A_i \left(\sum_{j=1}^{m+1} u_j\, \sigma_{i,j}\right), \quad (25)$$

where

$$d_i = \sum_{j=1}^{m+1} \rho_j\, \sigma_{i,j},\ i = 1, \cdots, m+1, \quad (26)$$

$$d_{m+2} = \sum_{i=1}^{m+1}\sum_{j=1}^{m+1} u_{i,j}\sigma_{i,j}, i < j, \quad (27)$$

$$d_{m+3} = \frac{1}{2}\sum_{i=1}^{m+1} u_{i,i}\, \sigma_{i,i}, \quad (28)$$

$$A_i = \sum_{j=1_{j\le k}}^{m+1}\sum_{k=1}^{m+1} \ell_{j,k,i} \times \begin{cases} \sigma_{j,k} & j = k, \\ 2\sigma_{j,k} & j < k, \end{cases} \quad (29)$$

For $(\theta_1, \cdots, \theta_m, \theta_{m+1}) \equiv (\beta_1, \cdots, \beta_m, \beta)$ and $u \equiv u(\beta_1, \cdots, \beta_m, \beta) = R_{\boldsymbol{s},\boldsymbol{k}}$, we obtain:

$$\rho_l = \frac{a_l - 1}{\beta_l} - b_l,\ l = 1, \cdots, m, \quad (30)$$

$$\rho_{m+1} = \frac{a_{m+1} - 1}{\beta} - b_{m+1}, \quad (31)$$

$$\ell_{l,l} = -\frac{nk_l}{\beta_l^2},\ l = 1, \cdots, m, \quad (32)$$

$$\ell_{m+1,m+1} = -\frac{n}{\beta^2}, \quad (33)$$

$$\ell_{l,k} = 0,\ l = 1, \cdots, m+1, l \neq k. \quad (34)$$

Using $\ell_{i,j}, i,j = 1, \cdots, m+1$, we can obtain $\sigma_{i,j}, i,j = 1, \cdots, m+1$ and

$$\ell_{l,l,l} = \frac{2nk_l}{\beta_l^3},\ l = 1, \cdots, m, \quad (35)$$

$$\ell_{m+1,m+1,m+1} = \frac{2n}{\beta^3}, \quad (36)$$

and other $\ell_{i,j,k} = 0$. Furthermore,

$$u_l = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{k_1-p_1}{q_1} \cdots \binom{k_m-p_m}{q_m} (-1)^{\sum_{l=1}^{m} q_l} \times \frac{\beta(p_l+q_l)}{\left(\sum_{l=1}^{m} \beta_l(p_l+q_l)+\beta\right)^2},\ l,k = 1, \cdots, m, \quad (37)$$

$$u_{m+1} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{k_1-p_1}{q_1} \cdots \binom{k_m-p_m}{q_m} (-1)^{\sum_{l=1}^{m} q_l} \times \frac{\sum_{l=1}^{m} \beta_l(p_l+q_l)}{\left(\sum_{l=1}^{m} \beta_l(p_l+q_l)+\beta\right)^2}, \quad (38)$$

$$u_{l,k} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{k_1-p_1}{q_1} \cdots \binom{k_m-p_m}{q_m} (-1)^{\sum_{l=1}^{m} q_l} \times \frac{2\beta(p_l+q_l)(p_k+q_k)}{\left(\sum_{l=1}^{m} \beta_l(p_l+q_l)+\beta\right)^3},\ l,k = 1, \cdots, m, \quad (39)$$

$$u_{m+1,m+1} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \times \cdots \binom{k_m}{p_m} \binom{k_1-p_1}{q_1} \cdots \binom{k_m-p_m}{q_m} (-1)^{\sum_{l=1}^{m} q_l+1} \times \frac{2\sum_{l=1}^{m} \beta_l(p_l+q_l)}{\left(\sum_{l=1}^{m} \beta_l(p_l+q_l)+\beta\right)^3}, \quad (40)$$

$$u_{l,m+1} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{k_1-p_1}{q_1} \cdots \binom{k_m-p_m}{q_m} (-1)^{\sum_{l=1}^{m} q_l+1} \times \frac{(p_l+q_l)\left(\sum_{l=1}^{m} \beta_l(p_l+q_l)-\beta\right)}{\left(\sum_{l=1}^{m} \beta_l(p_l+q_l)+\beta\right)^3},\ l = 1, \cdots, m. \quad (41)$$

After obtaining the above values from Eq. (25), $\hat{R}_{\boldsymbol{s},\boldsymbol{k}}^{Lin}$, Lindley's estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be obtained. All parameters should be computed at $\left(\hat{\beta}_1, \cdots, \hat{\beta}_m, \hat{\beta}\right)$, MLEs of $(\beta_1, \cdots, \beta_m, \beta)$.

4. Inference on $R_{s,k}$ in the General Case

If $X_1 \sim MWD(\beta_1, \gamma_1, \lambda_1)$, …, $X_m \sim MWD(\beta_m, \gamma_m, \lambda_m)$ and $Y \sim MWD(\beta, \gamma, \lambda)$ are independent random variables, then the MCSS parameter, $R_{\boldsymbol{s},\boldsymbol{k}}$, can be obtained from Eqs. (3) and (4) as:

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \dots \sum_{p_m=s_m}^{k_m} \binom{k_1}{p_1} \dots \binom{k_m}{p_m} \times \int_0^\infty \beta(\gamma + \lambda y)\, y^{\gamma-1} e^{\lambda y} e^{-\beta y^{\gamma} e^{\lambda y}} e^{-\left(\sum_{l=1}^m \beta_l p_l y^{\gamma_l} e^{\lambda_l y}\right)} \times \prod_{l=1}^m \left(1 - e^{-\beta_l y^{\gamma_l} e^{\lambda_l y}}\right)^{k_l - p_l} dy. \tag{42}$$

In this section, the Bayesian inference of $R_{\boldsymbol{s},\boldsymbol{k}}$ is studied under the squared error loss functions, where $\beta_1, \dots, \beta_m, \beta, \gamma_1, \dots, \gamma_m, \gamma, \lambda_1, \dots, \lambda_m$, and λ are independent gamma random variables:

$$\pi(\beta_l) \propto \beta_l^{a_l-1} e^{-b_l \beta_l}, a_l, b_l > 0, \quad l = 1, \dots, m, \tag{43}$$

$$\pi(\beta) \propto \beta^{a_{m+1}-1} e^{-b_{m+1}\beta}, a_{m+1}, b_{m+1} > 0, \quad l = 1, \dots, m, \tag{44}$$

$$\pi(\gamma_l) \propto \gamma^{c_l-1} e^{-d_l \gamma},\ c_l, d_l > 0, \quad l = 1, \dots, m, \tag{45}$$

$$\pi(\gamma) \propto \gamma^{c_{m+1}-1} e^{-d_{m+1}\gamma},\ c_{m+1}, d_{m+1} > 0, \tag{46}$$

$$\pi(\lambda_l) \propto \lambda_l^{e_l-1} e^{-f_l \lambda_l},\ e_l, f_l > 0, \quad l = 1, \dots, m, \tag{47}$$

$$\pi(\lambda) \propto \lambda^{e_{m+1}-1} e^{-f_{m+1}\lambda}, e_{m+1}, f_{m+1} > 0. \tag{48}$$

Similar to Section 2, since the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ cannot be evaluated in the closed form, it should be approximated using the MCMC method. From the joint posterior density function, the posterior PDFs of $\beta_1, \dots, \beta_m, \beta, \gamma_1, \dots, \gamma_m, \gamma, \lambda_1, \dots, \lambda_m$ and λ can be derived as follows:

$$\beta_l | \gamma, \lambda, \text{data} \sim \Gamma\left(nk_l + a_l, b_l + \sum_{i=1}^n \sum_{j=1}^{k_l} \left(R_j^{(l)} + 1\right) \times \left(x_{ij}^{(l)}\right)^{\gamma_l} e^{\lambda_l x_{ij}^{(l)}}\right),\ l = 1, \dots, m, \tag{49}$$

$$\beta | \gamma, \lambda, \text{data} \sim \Gamma\left(n + a_{m+1}, b_{m+1} + \sum_{i=1}^n (S_i + 1)\, y_i^{\gamma} e^{\lambda y_i}\right), \tag{50}$$

$$\pi(\gamma_l | \beta_l, \lambda_l, \text{data}) \propto \prod_{i=1}^n \prod_{j=1}^{k_l} \left(\left(x_{ij}^{(l)}\right)^{\gamma_l} \left(\gamma_l + \lambda_l x_{ij}^{(l)}\right) \right) \times \gamma_l^{c_l-1} e^{-d_l \gamma_l} \times e^{-\sum_{i=1}^n \sum_{j=1}^{k_l} \beta_l \left(R_j^{(l)}+1\right) \left(x_{ij}^{(l)}\right)^{\gamma_l} e^{\lambda_l x_{ij}^{(l)}}}, \quad l = 1, \dots, m, \tag{51}$$

$$\pi(\gamma | \beta, \lambda, \text{data}) \propto \prod_{i=1}^n \left(y_i^{\gamma} (\gamma + \lambda y_i) \right) \times e^{-\beta \sum_{i=1}^n (S_i+1) y_i^{\gamma} e^{\lambda y_i}} \gamma^{c_{m+1}-1} e^{-d_{m+1}\gamma}, \tag{52}$$

$$\pi(\lambda_l | \beta_l, \gamma_l, \text{data}) \propto \prod_{i=1}^n \prod_{j=1}^{k_l} \left(\gamma_l + \lambda_l x_{ij}^{(l)}\right) \times e^{-\sum_{i=1}^n \sum_{j=1}^{k_l} \beta_l \left(R_j^{(l)}+1\right) \left(x_{ij}^{(l)}\right)^{\gamma_l} e^{\lambda_l x_{ij}^{(l)}}} \lambda^{e_l-1} \times e^{-\lambda_l \left(f_l - \sum_{i=1}^n \sum_{j=1}^{k_l} x_{ij}^{(l)}\right)},\ l = 1, \dots, m, \tag{53}$$

$$\pi(\lambda | \beta, \gamma, \text{data}) \propto \prod_{i=1}^n (\gamma + \lambda y_i)\, e^{-\beta \sum_{i=1}^n (S_i+1) y_i^{\gamma} e^{\lambda y_i}} \times \lambda^{e_{m+1}-1} e^{-\lambda \left(f_{m+1} - \sum_{i=1}^n y_i\right)}, \tag{54}$$

As shown, samples should be generated from the posterior PDFs of γ_l and λ_l, $l = 1, \dots, m$, γ, and λ using the Metropolis-Hastings method as they are unknown PDFs. To this aim, the following Gibbs sampling algorithm is proposed:

1. Start by selecting an initial value $(\beta_{1(0)}, \dots, \beta_{m(0)}, \beta_{(0)}, \gamma_{1(0)}, \dots, \gamma_{m(0)}, \gamma_{(0)}, \lambda_{1(0)}, \dots, \lambda_{m(0)}, \lambda_{(0)})$.
2. Set $t = 1$.
3. m+2. Generate value $\gamma_{l(t)}$ from $\pi(\gamma_l | \beta_{l(t-1)}$, $\lambda_{l(t-1)}$,data) using the Metropolis-Hastings method, with $N\left(\gamma_{l(t-1)}, 1\right)$ as the proposal distribution.

m+3. Generate value $\gamma_{(t)}$ from $\pi(\gamma | \beta_{(t-1)}$, $\lambda_{(t-1)}$,data) using the Metropolis-Hastings method, with $N\left(\gamma_{(t-1)}, 1\right)$ as the proposal distribution.

m+4 - 2m+3. Generate value $\lambda_{l(t)}$ from $\pi(\lambda_l | \beta_{l(t-1)}$, $\gamma_{l(t-1)}$,data) using the Metropolis-Hastings method, with $N\left(\lambda_{l(t-1)}, 1\right)$ as the proposal distribution.

2m+4. Generate value $\lambda_{(t)}$ from $\pi(\lambda | \beta_{(t-1)}$, $\gamma_{(t-1)}$,data) using the Metropolis-Hastings method, with $N\left(\lambda_{(t-1)}, 1\right)$ as the proposal distribution.

2m+5 - 3m+4. Generate value $\beta_{l(t)}$ from $\Gamma\left(nk_l + a_l, b_l + \sum_{i=1}^n \sum_{j=1}^{k_l} \left(R_j^{(l)} + 1\right) \left(x_{ij}^{(l)}\right)^{\gamma_{l(t-1)}} e^{\lambda_{l(t-1)} x_{ij}^{(l)}}\right)$.

3m+5. Generate value $\beta_{(t)}$ from $\Gamma\left(n + a_{m+1}, b_{m+1} + \sum_{i=1}^n (S_i + 1)\, y_i^{\gamma(t-1)} e^{\lambda(t-1) y_i}\right)$.

3m+6. Evaluate

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \dots \sum_{p_m=s_m}^{k_m} \binom{k_1}{p_1} \dots \binom{k_m}{p_m} \times \int_0^\infty \beta_{(t)} \left(\gamma_{(t)} + \lambda_{(t)} y\right) y^{\gamma_{(t)}-1} e^{\lambda_{(t)} y} e^{-\beta_{(t)} y^{\gamma_{(t)}} e^{\lambda_{(t)} y}} \times e^{-\left(\sum_{l=1}^m \beta_{l(t)} p_{l(t)} y^{\gamma_{l(t)}} e^{\lambda_{l(t)} y}\right)} \times \prod_{l=1}^m \left(1 - e^{-\beta_{l(t)} y^{\gamma_{l(t)}} e^{\lambda_{l(t)} y}}\right)^{k_l - p_l} dy. \tag{55}$$

3m+7. Set $t = t + 1$.

3m+8. Repeat T times, steps 3 - 3m+7.

Therefore, the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ under the squared error loss functions is:

$$\hat{R}_{\boldsymbol{s},\boldsymbol{k}}^{MB} = \frac{1}{T} \sum_{t=1}^T R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{56}$$

Also, the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be constructed using the method proposed in [19].

5. Simulation Study and Data Analysis

5.1 Numerical Experiment and Discussion

This section compares different estimates using the Monte Carlo simulation. Point estimates are also compared with MSEs, and interval estimates are compared with AL and CP. Simulation studies were implemented using different censoring schemes, parameter values, and hyperparameters. The results are obtained based on 2000 repetitions, and the number of repetitions in the Gibbs sampling algorithm is $T = 3000$. Also, the significance level is set to 0.95 to obtain the HPD credible intervals. It is supposed that the simulated system has two strength components. Table 1 lists the censoring schemes used to get the results.

Three cases are considered. First, assuming the common parameters γ and λ are unknown, $(\beta_1, \beta_2, \beta_3, \beta, \gamma, \lambda) = (0.9,1.5,1,1.5,3)$ is used to obtain the simulation results. Also, two priors, namely Prior 1: $a_l = b_l = c_4 = d_4 = e_4 = f_4 = 0$, $l = 1, \ldots,4$ and Prior 2: $a_l = c_4 = e_4 = 0.3$, $b_l = d_4 = f_4 = 0.5$, $l = 1, \ldots,4$ are employed to compare the Bayes estimates of $R_{s,k}$ from Eq. (18). The simulation results are provided in Table 2. Second, assuming the common parameters γ and λ are known, $(\beta_1, \beta_2, \beta_3, \beta, \gamma, \lambda) = (1.5,2,1,1,2)$ is used to obtain the simulation results. Also, two priors, namely Prior 3: $a_l = b_l = c_4 = d_4 = e_4 = f_4 = 0$, $l = 1, \ldots,4$ and Prior 4: $a_l = c_4 = e_4 = 0.4$, $b_l = d_4 = f_4 = 0.8$, $l = 1, \ldots, 4$ are employed to compare the Bayes estimates of $R_{s,k}$. In this case, simulation results are obtained using Eqs. (22) and (25), which are provided in Table 3.

Table 1. Different censoring schemes

(k_l, K_l)		C.S	(n, N)		C.S
	R_1	(0,0,0,0,5)		S_1	(0,0,0,0,5)
(5,10)	R_2	(5,0,0,0)	(5,10)	S_2	(5,0,0,0)
	R_3	(1,1,1,1,1)		S_3	(1,1,1,1,1)
	R_4	$(0^{*9}, 10)$		S_4	$(0^{*9}, 10)$
(10,20)	R_5	$(10, 0^{*9})$	(10,20)	S_5	$(10, 0^{*9})$
	R_6	(1^{*10})		S_6	(1^{*10})

Table 2. Simulation results when common parameters γ and λ are unknown

		MCMC					
$(k_1, k_2, k_3, n, s_1, s_2, s_3)$	C.S	Prior 1			Prior 2		
		MSE	AL	CP	MSE	AL	CP
	(R_1, R_1, R_1, S_1)	0.0521	0.4925	0.937	0.0435	0.4625	0.942
(5,5,5,5,2,2,2)	(R_2, R_2, R_2, S_2)	0.0513	0.4815	0.938	0.0430	0.4651	0.943
	(R_3, R_3, R_3, S_3)	0.0510	0.4971	0.939	0.0428	0.4681	0.944
	(R_1, R_1, R_1, S_4)	0.0415	0.4025	0.943	0.0389	0.3625	0.947
(5,5,5,10,2,2,2)	(R_2, R_2, R_2, S_5)	0.0410	0.4053	0.944	0.0380	0.3641	0.948
	(R_3, R_3, R_3, S_6)	0.0423	0.4081	0.943	0.0375	0.3610	0.946
	(R_4, R_4, R_4, S_1)	0.0410	0.4112	0.944	0.0374	0.3636	0.948
(10,10,10,5,2,2,2)	(R_5, R_5, R_5, S_2)	0.0408	0.4151	0.942	0.0370	0.3719	0.946
	(R_6, R_6, R_6, S_3)	0.0415	0.4167	0.943	0.0367	0.3610	0.947
	(R_4, R_4, R_4, S_4)	0.0354	0.3562	0.948	0.0305	0.3025	0.952
(10,10,10,10,2,2,2)	(R_5, R_5, R_5, S_5)	0.0360	0.3526	0.949	0.0309	0.3074	0.951
	(R_6, R_6, R_6, S_6)	0.0349	0.3574	0.948	0.0307	0.3030	0.950
	(R_1, R_1, R_1, S_1)	0.0510	0.5017	0.939	0.0455	0.4671	0.942
(5,5,5,5,4,4,4)	(R_2, R_2, R_2, S_2)	0.0515	0.5061	0.938	0.0448	0.4788	0.940
	(R_3, R_3, R_3, S_3)	0.0518	0.5032	0.938	0.0469	0.4623	0.943
	(R_1, R_1, R_1, S_4)	0.0423	0.4152	0.944	0.0401	0.3777	0.949
(5,5,5,10,4,4,4)	(R_2, R_2, R_2, S_5)	0.0420	0.4185	0.942	0.0399	0.3721	0.947
	(R_3, R_3, R_3, S_6)	0.0410	0.4116	0.944	0.0409	0.3613	0.946
	(R_4, R_4, R_4, S_1)	0.0415	0.4184	0.943	0.0389	0.3514	0.946
(10,10,10,5, 4,4,4)	(R_5, R_5, R_5, S_2)	0.0413	0.4167	0.944	0.0380	0.3520	0.947
	(R_6, R_6, R_6, S_3)	0.0419	0.4225	0.944	0.0371	0.3535	0.948
	(R_4, R_4, R_4, S_4)	0.0348	0.3471	0.948	0.0300	0.3125	0.950
(10,10,10,10, 4,4,4)	(R_5, R_5, R_5, S_5)	0.0340	0.3625	0.949	0.0308	0.3085	0.951
	(R_6, R_6, R_6, S_6)	0.0352	0.3495	0.948	0.0310	0.3040	0.952

Table 3. Simulation results when common parameters γ and λ are known

		MCMC						Lindley	
$(k_1,k_2,k_3,n,s_1,s_2,s_3)$	C.S	Prior 3			Prior 4			Prior 3	Prior 4
		MSE	AL	CP	MSE	AL	CP	MSE	MSE
	(R_1,R_1,R_1,S_1)	0.0452	0.4725	0.937	0.0354	0.4570	0.942	0.0485	0.0436
(5,5,5,5,2,2,2)	(R_2,R_2,R_2,S_2)	0.0446	0.4731	0.938	0.0345	0.4532	0.942	0.0480	0.0439
	(R_3,R_3,R_3,S_3)	0.0459	0.4763	0.939	0.0340	0.4528	0.943	0.0496	0.0430
	(R_1,R_1,R_1,S_4)	0.0347	0.3855	0.940	0.0251	0.3526	0.947	0.0398	0.0325
(5,5,5,10,2,2,2)	(R_2,R_2,R_2,S_5)	0.0352	0.3896	0.941	0.0259	0.3544	0.948	0.0390	0.0320
	(R_3,R_3,R_3,S_6)	0.0340	0.3824	0.940	0.0240	0.3595	0.947	0.0400	0.0329
	(R_4,R_4,R_4,S_1)	0.0340	0.3842	0.942	0.0246	0.3580	0.949	0.0402	0.0330
(10,10,10,5,2,2,2)	(R_5,R_5,R_5,S_2)	0.0358	0.3985	0.940	0.0249	0.3560	0.949	0.0389	0.0328
	(R_6,R_6,R_6,S_3)	0.0350	0.3888	0.941	0.0256	0.3463	0.948	0.0389	0.0321
	(R_4,R_4,R_4,S_4)	0.0256	0.2645	0.948	0.0201	0.2050	0.950	0.0352	0.0245
(10,10,10,10,2,2,2)	(R_5,R_5,R_5,S_5)	0.0251	0.2633	0.949	0.0209	0.2036	0.951	0.0356	0.0240
	(R_6,R_6,R_6,S_6)	0.0260	0.2674	0.947	0.0200	0.2085	0.951	0.0349	0.0244
	(R_1,R_1,R_1,S_1)	0.0450	0.4730	0.938	0.0360	0.4566	0.940	0.0495	0.0436
(5,5,5,5,4,4,4)	(R_2,R_2,R_2,S_2)	0.0458	0.4749	0.939	0.0352	0.4585	0.943	0.0489	0.0438
	(R_3,R_3,R_3,S_3)	0.0462	0.4785	0.937	0.0349	0.4533	0.941	0.0496	0.0433
	(R_1,R_1,R_1,S_4)	0.0364	0.3896	0.942	0.0240	0.3582	0.948	0.0400	0.0330
(5,5,5,10,4,4,4)	(R_2,R_2,R_2,S_5)	0.0360	0.3845	0.943	0.0253	0.3564	0.947	0.0395	0.0334
	(R_3,R_3,R_3,S_6)	0.0357	0.3878	0.944	0.0261	0.3594	0.947	0.0396	0.0338
	(R_4,R_4,R_4,S_1)	0.0349	0.3823	0.942	0.0243	0.3436	0.948	0.0399	0.0329
(10,10,10,5, 4,4,4)	(R_5,R_5,R_5,S_2)	0.0340	0.3865	0.944	0.0260	0.3568	0.947	0.0397	0.0327
	(R_6,R_6,R_6,S_3)	0.0351	0.3799	0.943	0.0253	0.3463	0.948	0.0402	0.0331
	(R_4,R_4,R_4,S_4)	0.0250	0.2685	0.948	0.0208	0.2049	0.952	0.0350	0.0240
(10,10,10,10, 4,4,4)	(R_5,R_5,R_5,S_5)	0.0264	0.2631	0.947	0.0203	0.2066	0.951	0.0355	0.0247
	(R_6,R_6,R_6,S_6)	0.0260	0.2627	0.948	0.0207	0.2022	0.952	0.0359	0.0249

Table 4. Simulation results in general case

		MCMC					
$(k_1,k_2,k_3,n,s_1,s_2,s_3)$	C.S	Prior 5			Prior 6		
		MSE	AL	CP	MSE	AL	CP
	(R_1,R_1,R_1,S_1)	0.0573	0.5326	0.937	0.0485	0.5031	0.942
(5,5,5,5,2,2,2)	(R_2,R_2,R_2,S_2)	0.0586	0.5312	0.938	0.0480	0.5074	0.943
	(R_3,R_3,R_3,S_3)	0.0579	0.5347	0.939	0.0488	0.5066	0.944
	(R_1,R_1,R_1,S_4)	0.0467	0.4250	0.940	0.0375	0.3845	0.948
(5,5,5,10,2,2,2)	(R_2,R_2,R_2,S_5)	0.0469	0.4263	0.942	0.0379	0.3866	0.947
	(R_3,R_3,R_3,S_6)	0.0460	0.4227	0.942	0.0370	0.3825	0.947
	(R_4,R_4,R_4,S_1)	0.0469	0.4219	0.942	0.0368	0.3856	0.947
(10,10,10,5,2,2,2)	(R_5,R_5,R_5,S_2)	0.0472	0.4237	0.940	0.0379	0.3844	0.948
	(R_6,R_6,R_6,S_3)	0.0463	0.4230	0.944	0.0364	0.3829	0.946
	(R_4,R_4,R_4,S_4)	0.0407	0.3152	0.948	0.0289	0.2633	0.950
(10,10,10,10,2,2,2)	(R_5,R_5,R_5,S_5)	0.0400	0.3166	0.949	0.0280	0.2641	0.951
	(R_6,R_6,R_6,S_6)	0.0403	0.3147	0.948	0.0283	0.2677	0.952
	(R_1,R_1,R_1,S_1)	0.0570	0.5319	0.938	0.0489	0.5044	0.944
(5,5,5,5,4,4,4)	(R_2,R_2,R_2,S_2)	0.0579	0.5340	0.938	0.0486	0.5050	0.943
	(R_3,R_3,R_3,S_3)	0.0586	0.5362	0.939	0.0483	0.5033	0.942
	(R_1,R_1,R_1,S_4)	0.0475	0.4296	0.942	0.0374	0.3820	0.946
(5,5,5,10,4,4,4)	(R_2,R_2,R_2,S_5)	0.0470	0.4267	0.944	0.0377	0.3799	0.947
	(R_3,R_3,R_3,S_6)	0.0460	0.4318	0.943	0.0366	0.3866	0.948
	(R_4,R_4,R_4,S_1)	0.0479	0.4317	0.940	0.0369	0.3905	0.949
(10,10,10,5, 4,4,4)	(R_5,R_5,R_5,S_2)	0.0468	0.4268	0.940	0.0364	0.3822	0.948
	(R_6,R_6,R_6,S_3)	0.0473	0.4222	0.943	0.0364	0.3910	0.948
	(R_4,R_4,R_4,S_4)	0.0402	0.3188	0.946	0.0280	0.2799	0.951
(10,10,10,10, 4,4,4)	(R_5,R_5,R_5,S_5)	0.0408	0.3167	0.947	0.0281	0.2764	0.952
	(R_6,R_6,R_6,S_6)	0.0410	0.3200	0.947	0.0286	0.2602	0.952

Third, $(\beta_1, \beta_2, \beta_3, \beta, \gamma_1, \gamma_2, \gamma, \lambda_1, \lambda_2, \lambda) = (1, 2, 0.5, 1, 2, 1.5, 2, 1.5, 1)$ is used to obtain the simulation results. Also, two priors, namely Prior 5: $a_l = b_l = c_l = d_l = e_l = f_l = 0,\ l = 1, \ldots, 4$ and Prior 6: $a_l = c_l = e_l = 0.25,\ b_l = d_l = f_l = 0.45,\ l = 1, \ \ldots, 4$ are used to compare the Bayes estimates of $R_{s,\boldsymbol{k}}$ from Eq. (56), which are provided in Table 4.

Tables 2-4 show that the informative priors (priors 2, 4, and 6) perform best for the MSE values. Also, in the second case, the Bayes estimates obtained by the MCMC method serve better than the ones obtained by Lindley's approximation. It can also be observed that among the intervals, HPD intervals based on informative priors (priors 2, 4, and 6) performed best for the AL and CP values.

Furthermore, the following general results can be obtained from Tables 2-4:

- For fixed $\boldsymbol{s}$ and $\boldsymbol{k}$, MSEs and ALs decrease, and CPs increase by increasing n.
- For fixed $\boldsymbol{s}$ and n, MSEs and ALs decrease, and CPs increase by increasing $\boldsymbol{k}$.

The two items above may occur because the number of failures increases by increasing n, and consequently, more information is gathered, thereby improving the performance of estimates.

5.2 Real Data Analysis

This section analyzes a real dataset for illustrative aims. The data demonstrates strength measured in GPA for single carbon fibers. Single-fiber tensile tests were conducted at gauge lengths of 50 mm, 10 mm, and 1 mm, as found in [21]. This data type has recently been investigated in [22] as a stress-strength model for a two-parameter Rayleigh distribution. Suppose a system is composed of two different single-fiber gauge lengths such that the single fiber of 1 and 10 mm gauge lengths are considered strength, and that of 50 mm gauge length is the system's stress. Let X_1, X_2, and Y denote the single fiber with gauge lengths of 1 mm, 10 mm, and 50 mm, respectively. Thus, the X_1, X_2, and Y observations can be considered as follows:

$$\begin{bmatrix} 3.126 & 3.245 & 3.328 & 3.355 & 3.383 \\ 3.572 & 3.581 & 3.681 & 3.726 & 3.727 \\ 3.728 & 3.783 & 3.785 & 3.786 & 3.896 \\ 3.912 & 3.964 & 4.05 & 4.063 & 4.082 \\ 4.111 & 4.118 & 4.141 & 4.216 & 4.251 \\ 4.262 & 4.326 & 4.402 & 4.457 & 4.466 \\ 4.519 & 4.542 & 4.555 & 4.614 & 4.632 \\ 4.634 & 4.636 & 4.678 & 4.698 & 4.738 \\ 4.832 & 4.924 & 5.043 & 5.099 & 5.134 \\ 5.359 & 5.473 & 5.571 & 5.684 & 5.721 \end{bmatrix},$$

$$\begin{bmatrix} 2.35 & 2.361 & 2.396 & 2.397 & 2.445 \\ 2.454 & 2.454 & 2.474 & 2.518 & 2.522 \\ 2.525 & 2.532 & 2.575 & 2.614 & 2.616 \\ 2.618 & 2.624 & 2.659 & 2.675 & 2.738 \\ 2.74 & 2.856 & 2.917 & 2.928 & 2.937 \\ 2.937 & 2.977 & 2.996 & 3.03 & 3.125 \\ 3.139 & 3.145 & 3.22 & 3.223 & 3.235 \\ 3.243 & 3.264 & 3.272 & 3.294 & 3.332 \\ 3.346 & 3.377 & 3.408 & 3.435 & 3.493 \\ 3.501 & 3.537 & 3.554 & 3.562 & 3.628 \end{bmatrix}, \begin{bmatrix} 1.613 \\ 1.812 \\ 1.864 \\ 2.051 \\ 2.162 \\ 2.211 \\ 2.308 \\ 2.39 \\ 2.471 \\ 2.593 \end{bmatrix}$$

The data was normalized on a 0-1 scale to simplify calculations, which seemingly did not affect statistical inference. First, the MWD was fitted on three datasets separately, yielding the following results. For X_1, $(\beta, \gamma, \lambda) = (0.0032, 1.2614, 7.5423)$ and p-value $= 0.4565$. For X_2, $(\beta, \gamma, \lambda) = (5.2900e-04,\ 1.1672,\ 9.1953)$ and p-value$= 0.1999$. For Y, $(\beta, \gamma, \lambda) = (4.0904e-04, 1.1033, 9.3842)$ and p-value $= 0.9819$. From the p-values, it can be concluded that the MWD gave suitable fits for X_1, X_2, and Y datasets. The estimated parameters for different datasets show that they can be analyzed simply by considering the general case. Fig. 3 provides the empirical distribution functions and P-P plots for the three datasets above.

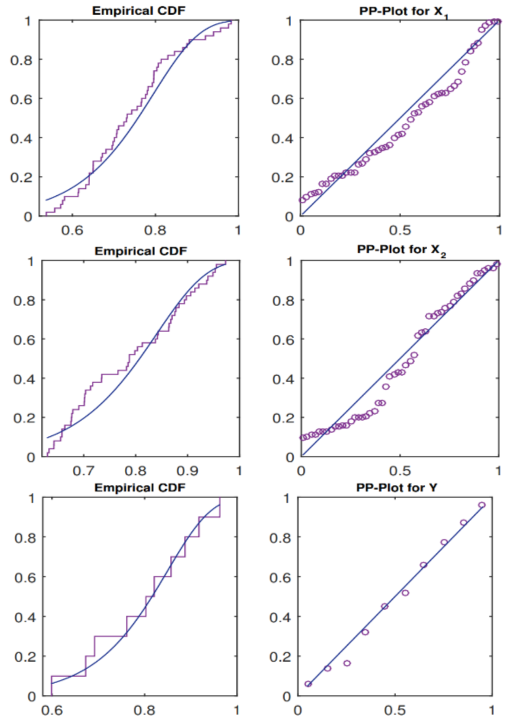


Figure 3. Empirical distribution function (left) and the PP-plot (right) for X_1 (first row), for X_2 (middle row), and for Y (third row)

For complete data set, putting $\boldsymbol{s} = (3,3)$ and $\boldsymbol{k} = (5,5)$ with non-informative priors yields $\hat{R}^{MB}_{\boldsymbol{s},\boldsymbol{k}}$ and the corresponding95% HPD interval by 0.2813 and $(0.1325, 0.5062)$, respectively. Accordingly, two different progressive censoring schemes can be generated as follows:

Scheme 1: $R^{\{(1)\}} = R^{\{(2)\}} = [0,0,1,0]$, $S = [1,1,1,0,0,0,0]$, $(\boldsymbol{k} = (4,4), \boldsymbol{s} = (2,2))$.
Scheme 2: $R^{\{(1)\}} = R^{\{(2)\}} = [0,1,1]$, $S = [2,1,1,1,0]$, $(\boldsymbol{k} = (3,3), \boldsymbol{s} = (1,1))$.

For Scheme 1 with non-informative priors, $\hat{R}_{s,k}^{MB}$ and the corresponding 95% HPD interval are obtained by 0.3051 and $(0.1536,0.5791)$, respectively. For Scheme 2 with non-informative priors, $\hat{R}_{s,k}^{MB}$ and the corresponding 95% HPD interval are obtained by 0.5012 and $(0.2035,0.8129)$, respectively. As we expected, a comparison between point and interval estimates indicates that Scheme 1 performs better than Scheme 2.

6. Conclusion

This paper considered the statistical inference of the MWD for an MCSS system with $m$non-identical component strengths in combination with a progressive censoring scheme. Bayesian point and interval estimates were considered when the standard parameters were unknown, known, and general. As $R_{s,k}$ and progressive censoring schemes could be converted to some cases, the problem solved in this paper is general.

Different estimates were compared using Monte Carlo simulations. The simulation results suggested that informative priors outperformed non-informative priors in point and interval estimates in Bayesian inference. Bayes estimates obtained by the MCMC method were superior to those obtained using Lindley's approximation. Besides, more information was gathered, and the accuracy of estimates increased by increasing the number of failures.

7. References

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