IJRRS
International Journal of Reliability, Risk Saftey: Theory and Application

Online ISSN: 2676-3346

Vol. 5/ Issue 1/ 2022/ pp. 69-75
DOI: 10.30699/IJRRS.5.1.9
Received: 2022.09.22, Accepted: 2022.10.15

Original Research Article

Nonparametric Estimation of the Family of Risk Measures Based on Progressive Type II Censored Data

Fatemeh Yousefzadeh [1*]

1. Department of Statistics, School of Mathematical Sciences and Statistic, University of Birjand, Birjand, Iran

*** fyousefzadeh@birjand.ac.ir**

Abstract

Tail risk analysis plays a central strategic role in risk management and focuses on the problem of risk measurement in the tail regions of extreme risks. As one crucial task in tail risk analysis for risk management, the measurement of tail risk variability is less addressed in the literature. Neither the theoretical results nor inference methods are fully developed, which results in the difficulty of modeling implementation. Practitioners are then short of measurement methods to understand and evaluate tail risks, even when they have large amounts of valuable data in hand. In this paper, some nonparametric methods of estimation for the class of variability measures among proportional hazards models based on progressively Type-II censored data are derived. We showed some properties of these estimators. Simulation studies have been performed to see the effectiveness of the proposed methods, and a real data set has been analyzed for illustrative purposes. Some well-known variability measures, such as the Gini mean difference, the Wang right tail deviation and the cumulative residual entropy, are, up to a scale factor, in this class.

Keywords: Cumulative residual entropy, Gini mean difference, Nonparametric estimation, Variability measure.

1. Introduction

In actuarial science, the study of large losses that occur with very small probability refersto the right tail risk analysis. In this framework, the value-at-risk (VaR), one of the more popular risk measures, is still widely used by insurance companies and financial institutions due to its conceptual simplicity. However, VaR has been criticized because it is not (in general) subadditive, and hence it is not coherent. An approach to solve this problem is to use the conditional tail expectation (TCE), which is the conditionalexpectation of the losses above the VaR. This measure is coherent and is the expectedsize of losses exceeding the VaR. Since tail events are subject to variability, the attention in the actuarial and financial literature is combining tail-loss measures (such as VaR and TCE). [1] combine TCE and different versions of tail variances into a single measure and derive explicit expressions in the framework of multivariate elliptical distributions. [2] consider additional tail variability measures to produce the so-called Gini shortfall. [3] obtain a new risk measure by combining TCE and the shortfall deviation. Following this approach, we estimate a wide class of variability measures based on distances among proportional hazards models. This family, besides including some of the previously cited measures, also contains other variability measures that sometimes perform better, such as Wang's right tail deviation ([4]) and the cumulative residual entropy ([5]). [6] constructed a distortion risk measure based on a loss variable and a benchmark variable in extreme scenarios to study the co-movement of the two variables. [7] extended the tail Gini functional of a univariate random variable to a case of a bivariate random vector as a tail risk variability measure that incorporates both the marginal risk severity of the loss variable and the tail dependence structure of the two variables. We study the nonparametric estimations for these measures.

Censoring schemes of statistical experiments occur naturally in survival and reliability and medical studies. Progressive Type II censoring scheme is one of the most generally popular censoring schemes. It has achieved great attention in the last decades. The progressive Type-II censoring scheme can be described as follows. Assume that n identical units $X_1, X_2, \ldots, X_n$ are located

on a life testing experiment and each unit has lifetime distribution with pdf f and cdf F. Suppose the vector of censoring scheme is $(m; R_1, R_2, \dots, R_m)$, where R_j demonstrates the number of units withdrawn from $n - \sum_{i=1}^{j-1} R_i - j$ surviving units in the j-th $(j = 1,2, \dots, m)$ stage of censoring and m is the prefixed number of removals. Based on this progressive censoring scheme, the following order statistics $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ are derived. In such a life-testing experiment, we successively perceive the failure units, but at the j-th stage of observation, R_j units are randomly chosen among the remaining units. In other words, the sample size is reduced progressively by $R_j + 1$ (one failed unit and R_j removed items). It is obvious that this type of censoring scheme contains the conventional type-II right censoring scheme $(R_1 = R_2 = \dots = R_{m-1} = 0, R_m = n - m)$ and the complete sampling scheme $(R_1 = R_2 = \dots = R_{m-1} = 0, n = m)$. More details about progressive censoring are may be located in [8] for more details about the progressive censoring. Suppose $F(X_{i:m:n}) = U_{i:m:n}$ be the i-th order statistic from uniform $U(0,1)$ distribution. Now, define

$V_1 = \frac{1-U_{m:m:n}}{1-U_{m-1:m:n}}$, $V_2 = \frac{1-U_{m-1:m:n}}{1-U_{m-2:m:n}}, \dots, V_m = 1 - U_{1:m:n}$

It is clear that V_i'is are independent random variables that follow $Beta\left(i + \sum_{j=m-i+1}^{m} R_j, 1\right)$. It can be seen that:

$U_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} V_j$

and

$E(U_{i:m:n}) = 1 - \prod_{j=m-i+1}^{m} \delta_j$

where $\delta_i = \frac{i+\sum_{j=m-i+1}^{m} R_j}{1+i+\sum_{j=m-i+1}^{m} R_j}$, $i = 1, \dots, m$ such that $\delta_j = \delta_1$ if $j \le 1$ and $\delta_j = \delta_m$ if $j \ge m$.

The rest of the paper is structured as follows. Different nonparametric methods of estimation and some theoretical properties for a wide class of variability measures among proportional hazards models based on Type-II progressive censoring are developed in Section 2. Simulation experiments, as well as analyses of real data, are performed in Section 3. Section 4 contains conclusions.

2. Nonparametric estimations of risk measures

[9] considered a non-negative random variable $\tilde{X}_{\alpha,\beta}$ defined by the condition

$$\left[\tilde{X}_{\alpha,\beta} | X_\beta = s\right] \underset{st}{\equiv} [X_\alpha - s | X_\alpha > s]$$

with survival function

$$\begin{aligned} \tilde{F}_{\alpha,\beta}(x) &= P\left(\tilde{X}_{\alpha,\beta} > x\right) \\ &= -\int_0^\infty \left(\frac{\bar{F}(x+s)}{\bar{F}(s)}\right)^\alpha d\bar{F}^\beta(s) \end{aligned}$$

We focus on estimating the family risk measure (FRM) $D_{\alpha,\beta}(X) = \frac{1}{\beta} E\left[\tilde{X}_{\alpha,\beta}\right]$, where $\alpha, \beta > 0$. This makes the measure symmetric with respect to the parameters.

Now, we have the class of measures $\Psi(x) = \left\{D_{\alpha,\beta}(X), \alpha, \beta > 0\right\}$.

Let X be a random variable with distribution function F. Given $\alpha, \beta > 0$, we have

$$D_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\alpha-\beta}\left(\int_0^\infty \bar{F}^\beta(x)dx - \int_0^\infty \bar{F}^\alpha(x)dx\right), & \alpha \neq \beta \\ -\int_0^\infty \bar{F}^\alpha(x)\log\bar{F}(x)dx, & \alpha = \beta. \end{cases} \quad (1)$$

Some particular members of this family have been recently considered in the literature. For example, for $\alpha = \beta = 1$, $D_{1,1}(X) = \xi(X)$ equals the cumulative residual entropy (CRE) proposed in reliability theory by [5], among others,[10],[11], and [12].

[13] proposed the classical Gini mean difference measure. Here for $\alpha = 1, \beta = 2$ we have

$$\begin{aligned} D_{1,2}(X) &= \frac{1}{2}E[|X - X'|] \\ &= \frac{1}{2}GMD(X), \end{aligned}$$

where $GMD(X)$ is the mean difference, which means the classical Gini mean difference $D_{1,2}(X) = \frac{1}{2}GMD(X)$.

For $\alpha = 1, \beta = 0.5$, we attain Wang deviation as $D_{1,0.5}(X) = 2W(X)$

where

$$W(X) = \int_0^\infty \sqrt{\bar{F}(x)}\, dx - \int_0^\infty \bar{F}(x)dx.$$

Generally, propounded $D_{\alpha,\beta}(X)$ with $\alpha = 1$ and $\beta > 0$, under the name of quantile-based dynamic cumulative residual Tsallis entropy of order α.

We develop various nonparametric estimates for $D_{\alpha,\beta}(X)$ quantity based on Type-II progressively censored samples. [9] represented $D_{\alpha,\beta}(X)$ as:

$$D_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\alpha-\beta}\int_0^1 F^{-1}(x)\left[\beta(1-x)^{\beta-1} - \alpha(1-x)^{\alpha-1}\right]dx, & \alpha \neq \beta \\ \int_0^1 F^{-1}(x)(1-x)^{\alpha-1}[1 + \alpha log(1-x)]dx, & \alpha = \beta. \end{cases} \quad (2)$$

Also, it can be easily seen that $D_{\alpha,\beta}(X, p)$ can be expressed as

$$D_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\alpha-\beta}\int_0^1 \left[(1-x)^\beta - (1-x)^\alpha\right] d\left(F^{-1}(x)\right), & \alpha \neq \beta \\ -\int_0^1 (1-x)^\alpha\, log(1-x) d\left(F^{-1}(x)\right), & \alpha = \beta. \end{cases} \quad (3)$$

2.1 Moment approximation method

In this method, the difference operator proposed by [15] for estimating the entropy is operated. This method was also used when estimating the extropy by [16] and [17]. This method is mainly based on the following fact:

$$\frac{d}{du}F^{-1}(u) \approx \frac{X_{i+w:m:n}-X_{i-w:m:n}}{F(X_{i+w:m:n})-F(X_{i-w:m:n})},$$

Where the window size $w \le m/2$, $X_{i:m:n} = X_{1:m:n}$ if $i < 1, X_{i:m:n} = X_{m:m:n}$ if $i > m$, and $X_{1:m:n} \le X_{2:m:n} \le \ldots \le X_{m:m:n}$ be a progressively Type-II censored sample of size m. It can be seen that (3) is approximately equal to:

$$D_{\alpha,\beta}(X) \approx \begin{cases} \frac{\frac{1}{(\alpha-\beta)m}\sum_{i=1}^{m}\left[\left(1-F(X_{i:m:n})\right)^{\beta}-\left(1-F(X_{i:m:n})\right)^{\alpha}\right]}{\frac{X_{i+w:m:n}-X_{i-w:m:n}}{F(X_{i+w:m:n})-F(X_{i-w:m:n})}}, & \alpha \neq \beta \\ \frac{-\frac{1}{m}\sum_{i=1}^{m}\left(1-F(X_{i:m:n})\right)^{\alpha} log(1-F(X_{i:m:n})}{\frac{X_{i+w:m:n}-X_{i-w:m:n}}{F(X_{i+w:m:n})-F(X_{i-w:m:n})}}, & \alpha = \beta. \end{cases}$$

Now, the moments-based estimate is proposed by replacing $U_{i+w:m:n} = F(X_{i+w:m:n})$ and $U_{i-w:m:n} = F(X_{i-w:m:n})$ by their expected values $E(U_{i:m:n})$ as given in [6]. This leads to the following estimate of $D_{\alpha,\beta}(X)$

$$\widehat{D}_{1,\alpha,\beta}(X) = \begin{cases} \frac{\frac{1}{(\alpha-\beta)m}\sum_{i=1}^{m}\left[\left(1-E(U_{i:m:n})\right)^{\beta}-\left(1-E(U_{i:m:n})\right)^{\alpha}\right]}{\frac{X_{i+w:m:n}-X_{i-w:m:n}}{E(U_{i+w:m:n})-E(U_{i-w:m:n})}}, & \alpha \neq \beta \\ \frac{-\frac{1}{m}\sum_{i=1}^{m}\left(1-E(U_{i:m:n})\right)^{\alpha} log\left(1-E(U_{i:m:n})\right)}{\frac{X_{i+w:m:n}-X_{i-w:m:n}}{E(U_{i+w:m:n})-E(U_{i-w:m:n})}}, & \alpha = \beta. \end{cases}$$

Based on [16], here, $D(X)$ is a variability measure if it satisfies the following intuitive properties:

1. $D(X + c) = D(X)$ for all constant c,
2. $D(cX) = cD(X)$ for all $c > 0$,
3. $D(c) = 0$ for any degenerate random variable at c,
4. $D(X) \ge 0$ for all risk X.

Proposition 21.

1. For all risk X, we have $\widehat{D}_{1,\alpha,\beta}(X) \ge 0$.
2. Let $Y = aX + b$, $a > 0$ then, $\widehat{D}_{1,\alpha,\beta}(y) = a\widehat{D}_{1,\alpha,\beta}(X)$.
3. $\widehat{D}_{1,\alpha,\beta}(X) = \widehat{D}_{1,\beta,\alpha}(X)$

Proposition 2.2. Under the progressive Type-II censoring scheme $R_1 = R_2 = \ldots = R_{m-1} = 0$, and $R_m = n - m$, which presents the conventional Type-II censoring at m. We have:

$$\widehat{D}_{1,\alpha,\beta}(X) = \begin{cases} \frac{\frac{1}{(\alpha-\beta)}\sum_{i=1}^{m}\left[\left(\frac{n-i+1}{n+1}\right)^{\beta}-\left(\frac{n-i+1}{n+1}\right)^{\alpha}\right]}{\frac{(n+1)(X_{i+w:m:n}-X_{i-w:m:n})}{2w}}, & \alpha \neq \beta \\ -\frac{\sum_{i=1}^{m}\left(\frac{n-i+1}{n+1}\right)^{\alpha} log\left(\frac{n-i+1}{n+1}\right)}{\frac{(n+1)(X_{i+w:m:n}-X_{i-w:m:n})}{2w}}, & \alpha = \beta. \end{cases}$$

Proof. It can be seen that under the conventional Type-II censoring with $R_1 = R_2 = \ldots = R_{m-1} = 0$, and $R_m = n - m$, $\boldsymbol{\delta_i} = \frac{i+n-m}{i+1+n-m}$ for all $i = 1, \ldots, m$. Now, by substituting this value in $\widehat{D}_{1,\alpha,\beta}(X)$, the result is obtained.

Proposition 2.3. $\widehat{D}_{1,\alpha,\beta}(X)$ is consistent estimator for $D_{\alpha,\beta}(X)$, i.e, $\widehat{D}_{1,\alpha,\beta}(X) \xrightarrow{p} D_{\alpha,\beta}(X)$ when $m \to n$, $n \to \infty$, $w \to \infty$, $\frac{w}{m} \to 0$.

Proof. Since when $m \to n$, the progressively Type-II censored sample becomes the complete sample. According to $\widehat{D}_{1,\alpha,\beta}(X)$ converges to the $D_{\alpha,\beta}(X)$ which were proved to be consistent.

2.2 Kernel-based method

Suppose that $\hat{f}(x)$ is estimated by kernel function K as follows:

$$\hat{f}(x) = \frac{1}{mh}\sum_{j=1}^{m} K\left(\frac{x-x_{j:m:n}}{h}\right),$$

Where $h > 0$ is a bandwidth, K is the kernel function that is a smooth and symmetric function that satisfies $\int k(x)dx = 1, \int xk(x)dx = 0$. Our estimate is proposed assuming the kernel function to be the standard normal density function

$$k(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and the bandwidth has chosen to equal, which attains the optimal smoothing formula, where m is the number of points. Now, we have the estimation of the survival function as

$$\hat{\bar{F}}(x) = \int_x^{\infty} \hat{f}(y)dy$$

therefore, the estimate of $D_{\alpha,\beta}(X)$ becomes

$$\widehat{D}_{2,\alpha,\beta}(X) = \begin{cases} \frac{1}{\alpha-\beta}\left(\int_0^{\infty} \hat{\bar{F}}^{\beta}(x)dx - \int_0^{\infty} \hat{\bar{F}}^{\alpha}(x)dx\right) & \alpha \neq \beta \\ -\int_0^{\infty} \hat{\bar{F}}^{\alpha}(x) log \hat{\bar{F}}(x)dx & \alpha = \beta \end{cases}$$

Where $\hat{F}(x) = \frac{1}{mh}\sum_{j=1}^{m} K\left(\frac{x-x_{j:m:n}}{h}\right)$, $K^{(x)} = \int_{-\infty}^{x} k\,(y)dy$.

Proposition 2.4.

1. $\widehat{D}_{2,\alpha,\beta}(X) \geq 0$ for all risk X.
2. suppose $Y = aX + b$, $a > 0$ then, $\widehat{D}_{2,\alpha,\beta}(y,p) = a\widehat{D}_{2,\alpha,\beta}(X)$.
3. $\widehat{D}_{2,\alpha,\beta}(X) = \widehat{D}_{2,\beta,\alpha}(X)$

Proof. The result can be obtained by using $\widehat{f_Y}(y) = \frac{1}{|a|}\widehat{f_X}(x)$.

Proposition 2.5. $\widehat{D}_{2,\alpha,\beta}(X)$ is a consistent estimator for $D_{\alpha,\beta}(X)$, i.e,

$\widehat{D}_{2,\alpha,\beta}(X) \xrightarrow{p} D_{\alpha,\beta}(X)$

when $m \to n$, $n \to \infty$, $w \to \infty$, $\frac{w}{m} \to 0$

Proof. The consistency of the kernel pdf, $\hat{f}(x) \to f(x)$ implies that

$\widehat{D}_{2,\alpha,\beta}(X) \xrightarrow{p} D_{\alpha,\beta}(X)$

2.3 Monte Carlo estimation method

The third estimator of $D_{\alpha,\beta}(X)$ (2) is proposed as follows based on Monte Carlo estimation

$$\widehat{D}_{3,\alpha,\beta}(X) = \begin{cases} \frac{1}{\alpha-\beta}\frac{1}{m}\sum_{i=1}^{m} X_{i:m:n}\left[\beta\hat{\hat{F}}^{\beta-1}(X_{i:m:n}) - \alpha\hat{\hat{F}}^{\alpha-1}(X_{i:m:n})\right] & \alpha \neq \beta \\ \frac{1}{m}\sum_{i=1}^{m} X_{i:m:n}\left(\hat{\hat{F}}(X_{i:m:n})^{\alpha-1}\right)\left[1 + \alpha log\hat{\hat{F}}(X_{i:m:n})\right] & \alpha = \beta \end{cases}$$

where $\hat{F}(X_{i:m:n}) = E(U_{i:m:n})$.

Proposition 2.6.

1. $\widehat{D}_{3,\alpha,\beta}(X) \geq 0$ for all risk X.
2. suppose $Y = aX + b$, $a > 0$ then, $\widehat{D}_{3,\alpha,\beta}(y,p) = a\widehat{D}_{3,\alpha,\beta}(X)$.
3. $\widehat{D}_{3,\alpha,\beta}(X) = \widehat{D}_{3,\beta,\alpha}(X)$

3. Simulation study

In this Section, we present some results of the Monte Carlo simulation study to assess the performances of the estimates for risk measures under different censoring schemes. We consider different censoring schemes in this study, including:

- One-Step from left with $(R_1 = n - m, R_2 = \ldots = R_m = 0)$.
- One-Step almost from the middle with $\left(R_1 = \ldots = R_{[m/2]-1} = 0, R_{[m/2]} = n - m, R_{[m/2]+1} = \ldots = R_m = 0\right)$.
- One-Step from right, the conventional Type-II censoring with $(R_1 = R_2 = \ldots = R_{m-1} = 0, R_m = n - m)$.
- Mixed with almost equal removals with $\left(R_1 = R_2 = \ldots = R_m \sim \frac{n-m}{m}\right)$

We compare the performances of the proposed estimates D_i's in terms of the mean square error (MSE) based on a Monte Carlo simulation study with 10000 iterations. Estimates depending on the window size w were computed assuming $w = \left[\sqrt{m} + 0.5\right]$, which was used for extropy estimates by [15]. For a particular n, m, and a censoring scheme, we generate a progressively censored sample from the exponential distribution $Exp(\theta)$ with $\theta = 1$. In each case, we compute all the estimates $\widehat{D}_{i,\alpha,\beta}$ $i = 1,2,3$ of the risk measures $D_{i,\alpha,\beta}$. We replicate the process 10000 times and compute the MSE associated with each estimate. It can be seen from Table 1 that risk measure estimates are affected, at different levels, by the sample size, censoring scheme, and the parent distribution of the data. As expected, when the sample size n increases, the MSE decreases for all estimates. For the exponential distribution, one can see from Table 1 that all risk measure estimates are satisfactory and perform almost the same under the MSE criterion except for D_3. It is clear that the kernel-based estimates produce high MSE values under one step from the right and mixed with equal removals censoring schemes and produces the lowest MSE values among all risk estimates almost under all censoring schemes except for one-step from left censoring schemes, in which D_1 produces slightly lower MSE values. Accordingly, one might observe that if the data comes from an exponential distribution, then D_1, and D_2 mostly perform better than the other estimates.

Table 1. Simulated risk measure estimator for the exponential distribution.

		$\beta = 0.5$			$\beta = 1$			$\beta = 2$		
	m, R	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3
10	5,R=(5,...,0)	0.99820	0.92517	1.12968	0.25821	0.25188	0.69677	0.08882	0.06956	0.071625
	5,R=(0,0,5,0,0)	1.1277	0.9941	1.2650	0.30532	0.28841	0.7622	0.08508	0.07163	0.09891
	5,R=(0,...,5)	1.2858	1.3593	1.3910	0.47128	0.63378	0.87573	0.07005	0.13921	0.34421
	5,R=(1,...,1)	1.20061	1.09873	1.53646	0.35338	0.41258	0.95858	0.06034	0.08522	0.25787
10	8,R=(2,...,0)	0.93746	0.82104	0.97279	0.19334	0.16926	0.63854	0.06766	0.04701	0.068690

		$\beta = 0.5$			$\beta = 1$			$\beta = 2$		
	m, R	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3
	8,R=(0,...,2,...,0)	1.08595	0.86840	1.06623	0.19014	0.17689	0.64914	0.05968	0.04506	0.059467
	8,R=(0,. ...,2)	1.17263	1.1249	1.37952	0.21730	0.31544	0.78829	0.03686	0.05965	0.30175
	8,R=(1,0,0,0,1,0,0,0)	1.05180	0.88044	0.98239	0.19202	0.17649	0.64640	0.06487	0.04449	0.092107
20	10,R=(10,0,...,0)	0.8926	0.7003	0.9178	0.1604	0.1426	0.2449	0.0521	0.0385	0.0725
	10,R=(0,...,10,...,0)	0.90980	0.83941	0.92429	0.20485	0.16174	0.28681	0.04374	0.03767	0.076127
	10,R=(0,...,10)	1.02294	0.96804	1.25204	0.47897	0.63046	0.65538	0.06583	0.13765	0.33145
	10,R=(1,...,1)	0.93604	0.74987	0.98345	0.30546	0.33566	0.55620	0.04167	0.06695	0.14378
20	17,R=(3,0,...,0)	0.59157	0.49126	0.63327	0.11884	0.08991	0.44615	0.03991	0.02394	0.08216
	17,R=(0,...,3,...,0)	0.74455	0.53724	0.80376	0.11119	0.09371	0.42873	0.03291	0.02168	0.099267
	17,R=(0,...,3)	0.9599	0.67845	0.94068	0.15362	0.24217	0.44123	0.01877	0.04016	0.046570
	17,R=(1,0,0,0,1,...,0)	0.68060	0.52209	0.93911	0.11467	0.09324	0.096598	0.03537	0.02271	0.045347
30	15,R=(15,0,...,0)	0.66519	0.53397	0.89533	0.12601	0.10074	0.23454	0.03848	0.02673	0.080107
	15,R=(0,...,15,...,0)	0.71184	0.69737	0.94298	0.17358	0.11889	0.25944	0.02973	0.02580	0.04246
	15,R=(0,...,15)	0.92059	0.88000	0.96844	0.47448	0.63240	0.71648	0.06362	0.13813	0.32830
	15,R=(1,,...,1)	0.91147	0.71660	0.95576	0.27749	0.30438	0.51542	0.03446	0.06153	0.07890
30	25,R=(5,0,...,0)	0.4581	0.3787	0.5394	0.09458	0.06445	0.27923	0.02964	0.01695	0.1512
	25,R=(0,...,5,...,0)	0.63057	0.42380	0.79130	0.07366	0.06692	0.075869	0.02220	0.01462	0.03399
	25,R=(0,...,5)	0.84902	0.77402	0.932764	0.15558	0.26522	0.27398	0.01518	0.04413	0.028371
	25,R=(1,0,0,0,1,0,...0)	0.60753	0.44449	0.74058	0.08916	0.06922	0.10841	0.02568	0.01368	0.06206

4. Numerical Example

To illustrate the ideas discussed above, we consider a data set presented by [17] (p. 156). The amounts presented in the following are inflation-adjusted (to 1981 using the U.S. Residential Construction Index) hurricane losses from 35 hurricanes that occurred between 1949 and 1980 and resulted in losses in excess of 5,000,000. The numbers shown are the amounts in excess of 5 million in units of 1,000. We treat the numbers in the following Tabas as the outcomes of the independent and identically distributed non-negative random variables $X_1, \ldots, X_{35}$. The data set is given by 1766 14030 42905 98217 356200 2123 20304 44397 118680 416680 5562 24112 47600 135136 508586 9474 25146 54917 187013 540778 10351 28727 58123 193446 745389 11983 35596 72809 222338 858881 13383 36409 97942 324511 1633000

The lognormal distribution adequately fits the data. When we use the lognormal model, the MLEs of the lognormal distribution parameters are $\hat{\mu} = 11.0447$ and $\hat{\sigma} = 1.6828$. Using the MLE of parameters, it follows that the MLEs of the risk measures of X are estimated depending on the complete data as

$\xi^{ml}(X) = D_{1,1}^{ml}(X) = 2.5161,\ GMD^{ml}(X) = 2D_{1,2}^{ml}(x) = 4.5105,\ W^{ml}(X) = 0.5D_{(1,.5)}^{ml}(x) = 1.2594.$

The risk measures are estimated in this section by using the proposed nonparametric estimates under various Type-II progressive censoring schemes with $m = 15$ assuming the window size $w = \lfloor\sqrt{15} + 0.5\rfloor = 4$. The following progressive Type-II censoring schemes are implemented in this example:

- Scheme 1:$(R_1 = 20, R_2 = \ldots = R_{15} = 0)$.
- Scheme 2: $(R_1 = \ldots = R_7 = 0, R_8 = 20, R_9 = \ldots = R_{15} = 0)$.
- Scheme 3: $(R_1 = \ldots = R_{14} = 0, R_{15} = 20)$.
- Scheme 4: $(R_1 = \ldots = R_5 = 1, R_6 = \ldots = R_{10} = 2, R_{11} = \ldots = R_{15} = 1)$.

The results for estimating the risk measures are obtained using the proposed estimates under the above-mentioned censoring schemes are summarized in Table 2. Table 2 shows that its results are consistent with the ones concluded from the simulation studies for moderate and large samples. That is, the kernel-based method performs

better than the other estimates when the data comes from Log-Normal distribution. Yet, other estimates also provide satisfactory results in some cases.

Table 2. risk measure estimator for n=35

	$\hat{\xi}_i$			$\widehat{GMD}_i$			$\hat{W}_i$		
m, R	1	2	3	1	2	3	1	2	3
Scheme1	2.44	2.48	2.33	4.908	4.474	4.315	1.24	1.25	1.25
Scheme2	2.43	2.44	2.26	4.63	4.62	4.23	1.22	1.23	1.32
Scheme3	2.61	2.41	2.10	4.72	4.93	4.84	1.16	1.21	1.42
Scheme4	2.54	2.43	2.59	4.34	4.36	4.60	1.39	1.37	1.52

5. Discussion and conclusions

In this paper, we have considered the estimation problem of the class of risk measures based on Type-II progressive censoring samples. Nonparametric-based methods involving moments approximation, Kernel-based, and monte Carlo methods have been discussed. It is evident that the best estimates of the risk measures depend on the parent model of the data, sample size, and censoring scheme. Generally, the estimates based on the kernel-based method compete with the other estimates in most of the considered cases. In our future work, we examine the nonparametric methods to investigate tail variability measure estimators in the case of multiple variables to capture the risk exposure caused by the co-movement of these financial variables. How to estimate tail variability while incorporating a tail dependence structure is a very challenging and interesting task.

6. References

[1] E. Valdez. "Tail conditional variance for elliptically contoured distributions". *Belgian Actuarial Bulletin,* vol. **5**, pp. 26–36,2005.

[2] E. Furman, R. Wang, R. Zitikis, "Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks." *Journal of Banking and Finance,* vol. **83**, pp.70–84, 2017.

[3] M.B. Righi, P.S. Ceretta, "Shortfall deviation risk: an alternative for risk measurement." *Journal of Risk,* vol.,**19**, pp. 81–116,2016

[4] S. Wang, S., "An actuarial index of the right-tail risk." *North American Actuarial Journal,* vol. **2**, pp. 88–101,1998.

[5] M. Rao, Y., Chen, B.C., Vemuri, and F. Wang, "Cumulative residual entropy: a new measure of information". *IEEE Transaction on Information Theory,* vol. **50**, pp.1220–1228,2004.

[6] Y. Hou, X. Wang," Nonparametric inference for distortion risk measures on tail regions. Insurance. "*Mathematics & Economics''* vol. 89, pp. 92–110. 2020.

[7] Y. Hou, X. Wang, "Extreme and inference for tail Gini functional with applications in tail risk measurement". *Journal of the American Statistical Association*vol.535, pp.1428–1443,2021

[8] N. Balakrishnan, R. Aggrawala. "Progressive censoring"*, theory, methods and applications.* Boston (MA), Birkhauser. 2000.

[9] G. Psarrakos, M.A., Sordo, "On a family of risk measures based on proportional hazards models and tail probabilities", *Insurance: Mathematics and Economics,* vol.**86**, pp.232–240,2019

[10] J. Navarro, Y. del Aguila, M., Asadi, "Some new results on the cumulative residual entropy". *Journal of Statistical Planning and Inference,* vol.**140**, pp.310–322,2010.

[11] M.A. Sordo, G. Psarrakos, "Stochastic comparisons of interfailure times under a relevation replacement policy". *Journal of Applied Probability,* vol.**54**, pp.134–145,2017.

[12] G. Psarrakos, A, Di Crescenzo, "A residual inaccuracy measure based on the relevation transform," *Metrika* vol.,**81**, pp.37–59,2018.

[13] R. Dorfman, R., "A Formula for the Gini Coefficient". *The Review of Economics and Statistics*vol.**61**, pp.146–149,1979.

[14] S.M. Sunoj, A.S. Krishnan P.G. Sankaran, "A quantile-based study of cumulative residual Tsallis entropy measures." *Physica A: Statistical Mechanics and its Applications,* vol.**494**, pp.410–421,2018.

[15] Q. Vasicek. "A test for normality based on sample entropy". *J R Stat Soc B*, vol. **38**, pp.54–59.1976.

[16] G. QiuK., Jia, "The residual extropy of order" statistics. *StatProbab Letter,* vol.**133**, pp.15–22. 2018a

[17] H.A. Noughabi J., Jarrahiferiz, "On the estimation of extropy". *J Nonparametr Statistics,* vol. **31**, pp.88–99,2019.

[18] P.J. Bickel, E.L. Lehmann, "*Descriptive statistics for nonparametric models IV*''. Spread In: Jureckova (Ed.), Contributions to Statistics. Reidel, Dordrecht,1979.

[19] S.A. Klugman, H.H., PANJER, and G.E. Willmot., "*Loss models: From data to decisions".* New York, John Wiley and Sons1998.