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Original Research Article

Reliability Evaluation of an Industrial System Using Lomax-Lindley Distribution

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Abstract

In this study, two parameters of the Lomax-Lindley distribution were developed, which generalized the existing Lindley distribution and has the growing and decreasing properties of the current distribution. The newly suggested Lomax-Lindley distribution parameters were estimated using maximum likelihood estimators (MLEs). Maximum likelihood estimators (MLEs) are biased for small and intermediate sample sizes. The two-parameter Lindley (TPL) distribution is increasingly being utilized to characterize data on lifetime and survival times because distribution can provide a better fit than several existing lifetime models. A real-world industrial system application is also provided to demonstrate how the concepts might be applied. The Mat Lab program was utilized for both the numerical result and the graphical representation.

Keywords: Lomax; Lindley; Reliability; Industrial system; Distribution; Sample size.

1. Introduction

A probability distribution called the Lindley distribution is used to simulate waiting times or durations in various contexts, including survival analysis, queuing systems, and reliability studies. It bears Dennis Lindley's name, a well-known Bayesian statistician. This distribution stands out for its ease of use and practicality in scenarios where more complex models might not be appropriate because of restrictions or limits. Unlike many other distributions, the Lindley distribution is highly flexible because it doesn't require strictly positive data. Typically, a distribution's probability mass function (PMF) or probability density function (PDF) involves a single parameter, commonly represented by δ, that governs the distribution's shape. When modeling non-negative integer-valued random variables or when zeros are included in the data, the Lindley distribution is frequently utilized.

Lindley [1] proposed the Lindley distribution with one parameter. The Lindley distribution with one parameter also combines the gamma $(2, \delta)$ and exponential (δ) distributions. The cumulative distribution function (cdf) and probability density function (pdf) of it are given by:

$$f(x;\delta) = \frac{\delta^2}{\delta+1}(1+x)e^{-\delta x}, \qquad x > 0, \delta > 0, \qquad (1)$$

$$F(x;\delta) = 1 - \left(1 + \frac{\delta x}{\delta+1}\right)e^{-\delta x}, \qquad x > 0, \delta > 0. \qquad (2)$$

Many scholars have recently examined the Lindley distribution as a lifetime model; for many years, it was eclipsed by the exponential distribution. Numerous statistical features, including moments, failure rate function, entropies, maximum likelihood (ML), stochastic ordering, and method of moments (MoM) estimations, were covered by Ghitany et al. [2]. They also demonstrated that the Lindley distribution can be a more accurate model than the exponential distribution using an actual data set. Mazucheli and Achcar [3] employed the

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Lindley distribution inside the basic competing risks distribution as a potential substitute for the Weibull or exponential distributions. Krishna and Kumar determined and examined the Lindley distribution's model features and reliability metrics [4]. They employed sample data that had been gradually Type-II censored for the estimating method. The estimation of the stress-strength parameter R=P(Y<X) for independent Lindley random variables X and Y was examined by Al-Mutairi et al. [5]. Particular academics have studied the parameter estimation of hybrid censored data with the Lindley distribution. See, for instance, Jia and Song [6], Maihulla et al. [7], Gupta and Singh [8], and Al-Zahrani and Ali [9]. Microbiological and organic fouling are two of the most common and challenging to manage types of foulants in surface water and wastewater applications [10]. Reliability analysis of solar photovoltaic systems Using Gumbel Hougaard Family Copula has been studied by [11].

In applied mathematics, reliability theory examines the dependability and non-reliability of processes, systems, and individual parts [9]. It focuses on determining how likely a system or component will function flawlessly for a specific time or under particular circumstances. The design, assessment, and enhancement of the dependability of diverse systems and products are accomplished through reliability theory in engineering, quality control, and risk analysis.

There are seven major sections in this article. Section 1 contained the introduction. Section 2 reviewed the relevant literature, while Section 3 covered the methodology. Section 4 provided the maximum likelihood estimation for the model's parameters. The application assessment and conclusion are presented in Sections 5 and 6, respectively.

2. Literature review

Yuan and Fuqing [12] investigated parameter estimation for bivariate Weibull distributions using the generalized moment approach for reliability evaluation. [13] developed a novel modified Weibull distribution function for assessing the strength of silicon carbide and alumina fibers. The exponentiated Weibull family is discussed in [14] for analyzing bathtub failure-rate data. [15] investigated a new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. [16] conducted research on an Improved new modified Weibull distribution: A Bayes investigation utilizing Hamiltonian Monte Carlo simulation. [17] Review the availability and reliability analysis for a system with a bivariate Weibull lifetime distribution. [18] investigated modifications to the Weibull distribution. [19] investigated a modified Weibull extension with a bathtub-shaped failure rate function. A Bayesian analysis based on Markov chain Monte Carlo simulation was studied by [20]. [21] presents a Bayes investigation of the reparameterized Weibull distribution using Hamiltonian Monte Carlo. [22] investigated the Bayesian inference and prediction of the inverse Weibull distribution for type II censored data numerically [23]. Demonstrated the application of the Gumbel-Hougaard family copula in solving the reliability modeling and performance evaluation of a solar photovoltaic system. [24] investigated the reliability, availability, maintainability, and dependability of photovoltaic systems. Reliability, availability, maintainability, and dependability analysis of a complex reverse osmosis machine system in water Purification was carried out [19]. However, none of the above work addressed the three subsystems with Marshall-Olkin bivariate Weibull distributions and processes, furthermore, as far as we know, in all the existing literature. However, in 2011, the [25, 26] investigated the arbitrary two-unit scheme. In this study, we investigated the three components of the water treatment system's reverse osmosis system. As the results indicate, all dependability matrices were investigated. It is evident from the body of current literature that no investigation has been conducted into the reliability studies of an industrial system employing a Weibull-Lindly distribution. Combining the two distributions yields a unique distribution that can fit any industrial system's data that can be repaired.

3. Methodology

The Lomax-Lindley distribution, or its enlarged versions, is employed in a systematic process of model selection, parameter estimates, reliability evaluation, validation, sensitivity analysis, and well-informed decision-making to examine and enhance the dependability of industrial systems.

The extended Lindley distribution is an expansion of the traditional Lindley distribution, which is frequently employed in dependability analysis. When evaluating or estimating an industrial system's reliability, choosing a suitable probability distribution is essential. The ability of the Lomax-Lindley distribution to accurately represent survival or failure times under specific situations led to its selection.

Using the based-line distribution for Shanker et al. [27], we created a unique probability distribution to simulate data from any complex industrial system.

These have been completed with the help of Alzaatreh et al. [28]. The statistical properties of the proposed distribution were motivated. Reliability analyses of the system have been evaluated. Both a numerical and graphical depiction are accessible.

The two-parameter Lindley (TPL) distribution, which Shanker et al. [27] suggested, has the following PDF and CDF:

$$f(x;\delta)=\frac{\delta^2}{\delta+\alpha}(1+\alpha x)e^{-\delta x},\ \ x>0,\delta>0,\ \alpha>-\delta \tag{3}$$

,

$$F(x;\delta)=1-\left(\frac{\delta+\alpha+\alpha\delta x}{\delta+\alpha}\right)e^{-\delta x},\quad x>0,\delta>0,\ \alpha>-\delta \tag{4}$$

The TPL distribution is the exponential distribution at α=0, and the one-parameter Lindley distribution at α=1.

The cumulative distribution function of the generalized distribution defined by Alzaatreh et al. [28] is given by:

$$G(x)=\int_0^{-\log[1-F(x)]} r(t)dt \tag{5}$$

The family of distributions defined by (5) is called the "Transformed – Transformer" family. The corresponding probability density function is given by:

$$g(x)=\frac{f(x)}{1-F(x)}r\{-\log(1-F(x))\} \tag{6}$$

The Weibull-X family CDF corresponds to the (6) above:

$$g(x)=\left(\frac{c}{\gamma}\right)\frac{f(x)}{1-F(x)}\left\{\frac{-\log(1-F(x))}{\gamma}\right\}^{(c-1)}, \tag{7}$$

where c is the shape parameter, and γ is the scale parameter.

Now, by substituting (3) and (4) into (7), we have

$$g(x)=\left(\frac{c}{\gamma}\right)\frac{\delta^2(1+\alpha x)}{(\delta+\alpha+\alpha\delta x)}\left\{\frac{\log\left(\left(\frac{\delta+\alpha+\alpha\delta x}{\delta+\alpha}\right)e^{-\delta x}\right)}{\gamma}\right\}^{(c-1)} \tag{8}$$

Where c and α are the shape parameters for the new proposed Weibull-Lomax distribution, and γ, is the scale parameters of the proposed distribution.

The corresponding derivative for the cumulative distribution function is;

$$\frac{d}{dx}\left(\left(\frac{c}{\gamma}\right)\frac{\delta^2(1+\alpha x)}{(\delta+\alpha+\alpha\delta x)}\left\{\frac{\log\left(\left(\frac{\delta+\alpha+\alpha\delta x}{\delta+\alpha}\right)e^{-\delta x}\right)}{\gamma}\right\}^{(c-1)} e^{\left\{-\left(\frac{\log\left(\left(\frac{\delta+\alpha+\alpha\delta x}{\delta+\alpha}\right)e^{-\delta x}\right)}{\gamma}\right)^{c}\right\}}\right) \tag{9}$$

let

$$u\,\frac{\delta+\alpha+\alpha\delta x}{\delta+\alpha}e^{-\delta x} \tag{10}$$

$$y=\frac{c}{\gamma}\frac{\delta^2(1+\alpha x)}{\delta+\alpha+\alpha\delta x}\left(\frac{\log(u)}{\gamma}\right)^{c-1}e^{-\left(\frac{\log(u)}{\gamma}\right)^{c}} \tag{11}$$

Then,

$$\frac{dy}{du}=\frac{d\left(\frac{c\delta^2(1+\alpha x)}{\gamma\delta+\alpha+\alpha\delta x}\left(\frac{\log(u)}{\gamma}\right)^{c-1}e^{-\left(\frac{\log(u)}{\gamma}\right)^{c}}\right)}{du} \tag{12}$$

$$\frac{dy}{du}=\frac{c^2}{\gamma^2}\frac{(c-1)\left(\frac{\log(u)}{\gamma}\right)\left(\frac{\log(u)}{\gamma}\right)^{c}-c\left(\frac{\log(u)}{\gamma}\right)^{c}e^{-\left(\frac{\log(u)}{\gamma}\right)^{c}}}{u} \tag{13}$$

$$\frac{du}{dx}=e^{y}\left(\frac{\alpha\delta e^{-\delta x}+(\delta+\alpha+\alpha\delta x)\delta e^{-\delta x}}{(\delta+\alpha+\alpha\delta x)^2}\right) \tag{14}$$

Products (13) and (14) give the resultant PDF of the newly proposed Weibull-Lomax distribution using the chain rule. The (15) below follows from (9).

$$F(x)=\frac{c^2}{\gamma^2}\frac{(c-1)\left(\frac{\log(u)}{\gamma}\right)\left(\frac{\log(u)}{\gamma}\right)^{c}-c\left(\frac{\log(u)}{\gamma}\right)^{c}e^{-\left(\frac{\log(u)}{\gamma}\right)^{c}}}{u}\,e^{y}\left(\frac{\alpha\delta e^{-\delta x}+(\delta+\alpha+\alpha\delta x)\delta e^{-\delta x}}{(\delta+\alpha+\alpha\delta x)^2}\right)$$

The reliability function corresponds to (15) above;

$$R(x)=1-\frac{c^2}{\gamma^2}\frac{(c-1)\left(\frac{\log(u)}{\gamma}\right)\left(\frac{\log(u)}{\gamma}\right)^{c}-c\left(\frac{\log(u)}{\gamma}\right)^{c}e^{-\left(\frac{\log(u)}{\gamma}\right)^{c}}}{u}\,e^{y}\left(\frac{\alpha\delta e^{-\delta x}+(\delta+\alpha+\alpha\delta x)\delta e^{-\delta x}}{(\delta+\alpha+\alpha\delta x)^2}\right) \tag{16}$$

The modified maximum likelihood estimators (MLEs), which are analytic second-order biases for the TPL distribution's parameters, are the subject of this article. Selecting an appropriate estimation technique is crucial for determining the parameters of any probability distribution. Because of its significant attractive qualities, the Maximum Likelihood Estimator (MLE) is the most used estimating approach. They are, for example, efficient, consistent, normally distributed, and asymptotically unbiased. Nonetheless, it should be highlighted that most of these characteristics rely on the large sample size requirement. For small and intermediate sample sizes, it may not be appropriate, particularly regarding the unbiasedness property. Because of this, it is crucial to create virtually unbiased estimators for the TPL distribution.

Finding nearly unbiased estimators for the Lomax-Lindley's parameters is the primary goal of this investigation. Furthermore, an actual data application is showcased to illustrate the methodologies' suitability and the application of the proposed model to the industrial system.

4. Maximum Likelihood Estimation

Suppose that $\alpha=(\alpha_1,\dots,\alpha_n)$ be a random sample of size n from the Lomax-Lindley distribution with pdf (1). The log-likelihood function is

$$l(\boldsymbol{\delta}|\alpha)=\gamma log(\delta^2)-\gamma log(\delta+\alpha)+\sum_{x=1}^{\gamma}log(1+\alpha c)-\delta\sum_{x=1}^{\gamma}\alpha_i\ , \tag{17}$$

where $\boldsymbol{\delta}=(\boldsymbol{\delta},\mathrm{c})$. The MLEs of $\hat{\theta}$ and $\hat{c}$ of the unknown parameters θ and α are obtained by solving the non-linear equations

$$\frac{\partial l}{\partial\delta}=\frac{2\gamma}{\delta}-\frac{c}{\alpha+\delta}-\sum_{x=1}^{\gamma}\alpha_i\ , \tag{18}$$

$$\frac{\partial l}{\partial c}=-\frac{\gamma}{c+\delta}+\gamma\sum_{x=1}^{\gamma}(y_i)(1+c\alpha)^{-1} \tag{19}$$

The Equations (18) and (19) do not seem to be solved directly. Therefore, a suitable numerical algorithm must be used. The use of Maple software for further numerical evaluation is necessary.

5. Application

Here, we demonstrate the Lindley-Lomax model's applicability. We fit the Lindley-Lomax distribution to the industrial reverse osmosis system data set using the Lindley-Lomax models. Table 2, which corresponds to Fig. 1 below, shows the failure rate value analysis over a 12-month period and the associated time in months.

The system was assessed by determining the sample size γ =5, 10, and 15, δ=0.5, 1.0, 1.5, 2.0 and 2.5, c=0.5, 1.0, 1.5 and 3.0. The pseudo-random samples are simulated using the inverse transform method from TPL distribution, that is $x = (x_1, \dots, x_\gamma)$ is generated from:

Table 1. Estimation for δ and α, ($\alpha = 1.5$) using MLE

δ	γ	MLE
1.0	10	0.124
	30	0.054
	50	0.041
1.5	10	0.254
	30	0.043
	50	0.057
2.0	10	0.483
	30	0.556
	50	0.056
3.0	10	0.342
	30	0.274
	50	0.142
5.0	10	0.451
	30	0.231
	50	0.183

Table 2: Weibull-Lindly distribution's reliability of an industrial system with time

Time↓ $\boldsymbol{\delta}$→	0.0001	0.0002	0.0003	0.0004
0	0.9999	0.9998	0.9997	0.9996
10	0.8912	0.8815	0.8834	0.8645
20	0.7799	0.7768	0.7752	0.7565
30	0.6724	0.6658	0.6686	0.6445
40	0.5655	0.5545	0.5505	0.5375
50	0.4587	0.4436	0.4395	0.4238
60	0.3413	0.2348	0.3265	0.3079
70	0.2386	0.1277	0.2137	0.2938
80	0.1287	0.0175	0.1069	0.1879
90	0.0145	0.0124	0.0937	0.0727
100	0.0124	0.0119	0.0093	0.0068
330	0.0095	0.0086	0.0075	0.0047
360	0.0079	0.0071	0.0049	0.0026

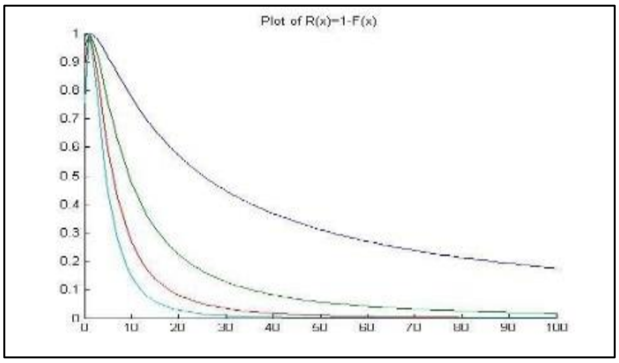

Figure 1. Reliability of an industrial system

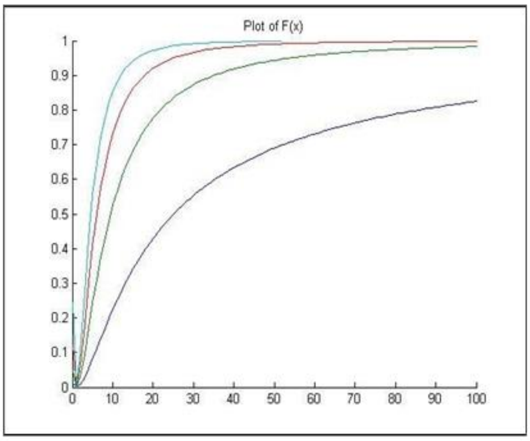

Figure 2. Probability density function for the proposed distribution

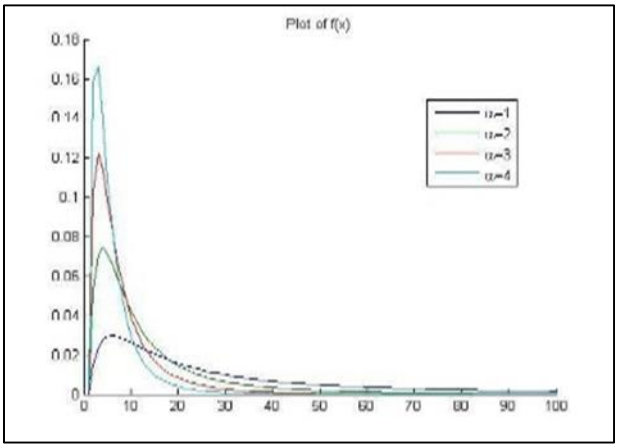

Figure 3. Cumulative distribution function for the proposed distribution

6. Conclusions

Over the past ten years, scholars have become increasingly interested in creating new extended models from traditional ones. This study presents the "Lindley-G" family of distributions, a new class of models capable of generating all classical continuous distributions. We define matching Lindley G distribution for any parent continuous distribution G.

The Lindley-Lomax distribution, a novel extension of the Lomax distribution, was proposed in this study. It suggested a few mathematical characteristics of the distribution, including its characteristic functions, moment-generating functions, and ordinary moments. Along with the distribution of ordered statistics of the new distribution, the reliability analysis of the distribution, which included the survival and hazard functions, was also considered. The distribution is skewed and flexible, as specific data visualizations showed. The maximum likelihood estimation method has been used to estimate the model parameters. The plots' implications for the survival function suggest that time-or age-dependent events, where the survival rate falls with increasing age, could be modeled using the Lindley-Lomax distribution. Applying the model to a real-world dataset on the reliability analysis of repairable industrial systems illustrates the performance of the new distribution. The study of the data revealed that the latest distribution, the Lindley-Lomax distribution, outperforms the transmuted Power Lomax distribution, the Weibull-Lomax distribution, the Power Lomax distribution, and the traditional Lomax distribution. Our model's success suggests that the suggested model can well describe further real-world scenarios.

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7.2 Conflict of interest

The authors declare that they have no conflicts of interest.

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